Teacher Assignment and Minimal Envy Mechanisms*

Julien Combe†

March 30, 2018

Abstract

We investigate the teacher assignment problem where one has to assign both newly tenured teachers without an initial assignment and tenured teachers with an initial position. Each school has a priority ordering over the teachers. In this framework, there is no mechanism that is both i) individually rational (IR), i.e., that assigns each teacher to a school that he/she weakly prefers to his/her initial one, and ii) stable, i.e., there are no blocking pairs: teachers and schools that are not assigned together and would prefer to be. A mechanism has minimal envy if there is no other mechanism that always leads to a set of blocking pairs included in the one of the former mechanism. We show that the modified Deferred Acceptance mechanism (DA∗) is a minimal envy mechanism in the set of IR and strategy-proof mechanisms. We also show that, in a one-to-one setting, an extension of the Top Trading Cycle (TTC∗) mechanism is also a minimal envy mechanism in the set of IR, strategy-proof and Pareto-efficient mechanisms. These results generalize the results in the school choice literature concerning the standard DA and TTC mechanisms.

JEL Classification: C78, D47, D63.

Keywords: Matching, Teacher assignment, Reassignment, Minimal envy mechanisms.

---

*I am grateful to Elias Bouacida, Yeon-Koo Che, Simon Gleyze, Philippe Jehiel, Josué Ortega, Vasiliki Skreta, Olivier Tercieux and seminar participants at UCL. All potential mistakes are mine.

†University College London, United Kingdom. Email: j.combe@ucl.ac.uk.
1 Introduction

Since the seminal work of Gale and Shapley (1962), the study of matching problems without monetary transfers has become increasingly important, and solutions to these problems have been widely used in many practical applications, such as the allocation of interns to hospitals (see, for instance, Roth, 1984), school choice (Abdulkadiroglu and Sonmez, 2003), kidney exchange (Roth, Sonmez, and Unver, 2004) and the reallocation of houses (Shapley and Scarf, 1974). Some authors have recently investigated the problem of reassigning agents with an initial allocation who have preferences over objects and where the objects have priorities over the agents. Many applications have been identified for such problems: the (re)assignment of campus housing (Guillen and Kesten, 2012), the reallocation of workers to positions (Compte and Jehiel, 2008) and the assignment of teachers to schools (Pereyra, 2013; Combe, Tercieux, and Terrier, 2016).

In the reallocation of houses, called housing market problems, there is a set of individuals, and each individual is initially assigned to a house. Individuals have preferences over the houses, so there can be beneficial exchanges between them. In such a setting, two important properties have been identified. First, each individual has the right to stay in his/her initial house, so any new matching of individuals to houses must allocate each individual to a house that he/she weakly prefers to his/her initial one. This property is called individual rationality. Second, a matching is Pareto-efficient if one cannot find another matching such that all the individuals weakly (and some strictly) prefer their new assignment. The problem is to find a systematic procedure, called a mechanism, that always returns an individually rational and Pareto-efficient matching for any report of the preferences of the individuals. Moreover, an attractive property for such a mechanism is to give incentives to the agents to truthfully report their preferences: such mechanisms are called strategy-proof mechanisms.\(^1\) Shapley and Scarf (1974) proposed an algorithm, the Top Trading Cycle (TTC) mechanism that is individually rational, Pareto-efficient and strategy-proof. For the case where some agents do not have any initial house and some houses are possibly vacant, Abdulkadiroglu and Sonmez (1999) proposed a strategy-proof and Pareto-efficient mechanism called the You-Request-My-House-I-Get-Your-Turn (YRMH-IGYT) algorithm. Karakaya, Klaus, and Schlegel (2017) extended the latter to a generalized TTC mechanism with priorities\(^2\) (which we refer to as TTC\(^*\)), and showed that it is the only IR, strategy-proof, Pareto-efficient, consistent and reallocation-proof mechanism.\(^3\)

In school choice problems, one has to assign students to schools. The former have preferences over the schools, and each school has a priority ordering over the students. Contrary to a housing market problem, students do not have any initial school and each school has a priority ordering over the students. In such a setting, Gale and Shapley (1962), and later on Abdulkadiroglu and Sonmez (2003), identified important properties for matching. A stable or envy-free matching is a matching where there is no blocking pair, i.e., a student and a school who are not assigned together but would prefer to be. Similarly to the housing market problem, a Pareto-efficient matching does not admit another matching such that all the students are weakly better off and some strictly. Such properties are incompatible; thus, two important mechanisms have been identified:

- The Deferred-Acceptance (DA) mechanism proposed by Gale and Shapley (1962), which is stable

---

1Formally, it is a dominant strategy for individuals to report their true preferences. We refer the reader to the formal definition in Section 2.

2It can also be viewed as an extension of the school choice TTC mechanism introduced below. Basically, Abdulkadiroglu and Sonmez (1999)’s mechanism is equivalent to the Serial Dictatorship (SD) mechanism when there are only unassigned agents and vacant houses. In the same setting, Karakaya, Klaus, and Schlegel (2017)’s mechanism is equivalent to the school choice TTC mechanism. If the houses all have the same priorities, then TTC is equivalent to SD.

3For the latter two properties, we refer the reader to cited article for further details. Intuitively, consistency requires that the mechanism still chooses the same allocation if one removes (consistently) some agents with their assigned house. Reallocation-proofness requires that no two agents can jointly misreport their preferences, exchange their allocation afterward and be better off than when being truthful.


and strategy-proof (for the students). They showed that the DA mechanism is constrained efficient: all the students weakly prefer their matching to any other stable matching.

- The school choice Top Trading Cycle mechanism, proposed by Abdulkadiroglu and Sonmez (2003), which incorporates the priority of the schools and is Pareto-efficient and strategy-proof.

Pareto-efficient mechanisms are not stable. Thus, a natural question is how to compare Pareto-efficient mechanisms in terms of “how much envy” they generate. Abdulkadiroglu, Che, Pathak, Roth, and Tercieux (2017) introduced the concept of minimal envy. The idea is to compare mechanisms according to the sets of blocking pairs that they generate for each preference and priority profile. If one mechanism always leads to a set of blocking pairs included in another, and this inclusion is strict for some profiles, then we say that the former has strictly less envy than the latter. A mechanism is a minimal envy mechanism if there is no other mechanism that has strictly less envy. In a one-to-one framework where schools only have one seat, they showed that in a school choice setting, TTC is a minimal envy mechanism in the set of strategy-proof and Pareto-efficient mechanisms. Last, since DA is a stable mechanism, its set of blocking pairs is empty in every preference and priority profile, so it is trivially a minimal envy mechanism in the set of strategy-proof mechanisms.

Teacher assignment problems can be seen as a hybrid setting between the two aforementioned models. When there are only newly tenured teachers, we return to a standard school choice model, and when there are only tenured teachers, we are in a housing market model where houses have priorities. Similarly to school choice problems, one has to assign teachers to schools where the former have preferences over the latter and where each school has a priority ordering over the teachers so that stability is an appealing property. Newly tenured teachers do not have any initial assignment but, as in housing market problems, tenured teachers are initially assigned to a school and are looking for a reassignment. Thus, granting the latter the right to stay at their initial school, i.e., individual rationality, is also an important property. Guillen and Kesten (2012) first studied this problem in the context of the assignment of the students to campus housing at MIT. In this framework, although stability and individual rationality appear to be desirable properties, they are incompatible: if one wants to ensure individual rationality, blocking pairs will inevitably arise. Our results show that one can generalize the minimal envy results of DA and TTC in the school choice setting to their two-counterpart mechanisms in a teacher assignment framework: i) the modified Deferred Acceptance (DA∗) identified by Guillen and Kesten (2012) and studied by Pereyra (2013) and ii) the generalized Top Trading Cycle (TTC∗) mechanism of Karakaya, Klaus, and Schlegel (2017). Indeed, one can see our results as a generalization since when there are only newly tenured teachers, the model becomes a standard school choice problem and the two mechanisms collapse to the standard DA and TTC. Table 1 summarizes our findings in a teacher assignment setting and compares them to their equivalent results in the literature of school choice. As mentioned, in a school choice setting, DA being stable, it is trivially a minimal envy mechanism in the set of strategy-proof mechanisms. However, in a teacher assignment setting, the same result for its counterpart DA∗ is not trivial since the latter mechanism is not stable. Thus, part of our contribution is to tackle this difficulty. For TTC∗, our proof extends the technique of Abdulkadiroglu, Che, Pathak, Roth, and Tercieux (2017) to the case where tenured teachers are present.

Guillen and Kesten (2012) identified that the mechanism used at MIT, the NH4 algorithm, is a DA mechanism in which each student is systematically ranked at the top of the priority ordering of its initial house. In doing so, one can ensure that the resulting matching is individually rational. However, when assigning teachers to schools, for instance, the priorities of the schools are often determined by law so

---

4Due to the applications that we focus on, we refer to the version of the algorithm where the students propose and assume that schools are not strategic in their reports. We refer the reader to Section 3, which formally introduces the mechanism. The strategy-proofness of the mechanism was proved by Roth (1982) and Dubins and Freedman (1981).

5However, note that DA is not a Pareto-efficient mechanism, so it does not contradict the result concerning TTC since the latter focuses on the set of strategy-proof and Pareto-efficient mechanisms.
<table>
<thead>
<tr>
<th>Setting</th>
<th>School Choice</th>
<th>Teacher Assignment</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mechanism</strong></td>
<td>DA</td>
<td>DA$^*$</td>
</tr>
<tr>
<td><strong>Schools’ capacities</strong></td>
<td>Multiple seats</td>
<td>One seat</td>
</tr>
<tr>
<td><strong>Minimal envy in the set</strong></td>
<td>SP</td>
<td>SP &amp; PE</td>
</tr>
<tr>
<td></td>
<td></td>
<td>IR &amp; SP</td>
</tr>
<tr>
<td></td>
<td></td>
<td>IR &amp; SP &amp; PE</td>
</tr>
</tbody>
</table>

Table 1: Summary of the results

† Notes: The second line states whether the minimal envy result holds in a many-to-one environment (Multiple seats) or a one-to-one environment (One seat). The last line describes the set of mechanisms in which the result holds. SP refers to strategy-proof, PE to Pareto-efficient, IR to individually rational.

that teachers are not necessarily the highest ranked in the priority ordering of their initial school. For instance, in France, a point-based system is used, where teachers accumulate points based on several characteristics, such as experience, teaching in a deprived area, and whether their partner lives in a different region. Schools then rank teachers in decreasing order of points. These criteria do not ensure that teachers have the highest number of points in their initial school.\footnote{We refer the reader to Combe, Tercieux, and Terrier (2016) for a description of the institutional details or to the website of Matching in Practice: http://www.matching-in-practice.eu/matching-practices-of-teachers-to-schools-france/.

7 The matchings respecting this notion of stability were called $\mu_0$-stable matchings by Compte and Jehiel (2008) and acceptable matchings by Pereyra (2013).

8 See, for instance, the textbook of Roth and Sotomayor (1990).}

Therefore, even when schools do not first rank their initial teachers, a natural modification of the DA mechanism can be used to ensure that the resulting matching is individually rational: first, artificially modify the original priorities of the schools so that they move their initial teachers to the top of their ordering and then run the standard DA mechanism over these modified priorities. This mechanism, called DA$^*$, is by construction, individually rational and strategy-proof. Compte and Jehiel (2008) and Pereyra (2013) studied its properties. In particular, while not being stable with respect to the original priorities, this mechanism respects a weaker form of stability. If a teacher prefers the school assigned to another teacher with a lower priority than him/her, then this school must be the initial school of the latter. In other words, if a teacher prefers a school to his/her current school under DA$^*$, then any teacher assigned to it and who was not initially assigned to it must have a higher priority than him/her according to the original priorities.\footnote{Combe, Tercieux, and Terrier (2016) showed that DA$^*$ suffers from an important drawback, it is not two-sided Pareto-efficient (2-PE), i.e., one can reassign teachers such that both teachers and schools obtain weakly better (and some strictly better) matches. In school choice models, it is well known\footnote{See, for instance, the textbook of Roth and Sotomayor (1990).} that stable matchings are 2-PE. They identify a mechanism, the Teacher Optimal Block Exchange (TO-BE) mechanism, which is individually rational for teachers and the schools (2-IR), 2-PE and strategy-proof. In particular, this mechanism can, for some preference and priority profiles, return a matching that has strictly less envy than DA$^*$, i.e., has a set of blocking pairs strictly included in the one of DA$^*$. However, for some other profiles, the reverse can occur such that TO-BE does not have strictly less envy than DA$^*$ or vice-versa. Thus, the question whether one can find an IR and SP mechanism that has strictly less envy than DA$^*$ has remained an open question, and our results show that it is indeed not possible.

Karakaya, Klaus, and Schlegel (2017) introduced a generalized version of the YRMH-IGYT mechanism proposed by Abdulkadiroglu and Sonmez (1999). One can easily adapt their mechanism to a (one-to-one) teacher assignment setting; we refer to it as TTC$^*$. If there are only newly tenured teachers without any initial assignment, the mechanism becomes equivalent to the school choice TTC; thus, Abdulkadiroglu, Che, Pathak, Roth, and Tercieux (2017) showed that it is a minimal envy mechanism in the set of strategy-proof and Pareto-efficient mechanisms. If there are only tenured teachers with an initial school, it becomes the standard TTC algorithm for housing market problems, and Ma (1994) showed that it is the only mechanism that is individually rational, strategy-proof and Pareto-efficient, so in this singleton set, it is trivially a minimal envy mechanism. For an intermediate model with a combination of newly
tenured and tenured teachers, the set of IR, SP and PE mechanisms is not a singleton, but we show that the result of Abdulkadiroglu, Che, Pathak, Roth, and Tercieux (2017) can be extended to this setting.

In Section 2, we start to formally introduce the teacher assignment setting. Then, in Section 3, we define the modified DA (DA*) and generalized TTC (TTC*) and, for each, precisely discuss the links in the literature. In Section 4, we provide our results and the intuitions of the proofs through an example. The proof concerning DA* being of interest in itself, we relegate it to Section 5. Finally, in Section 6, we discuss possible alternative mechanisms and open questions for future research before concluding in Section 7.

2 Setting

Consider a teacher assignment problem where there is a finite set $T$ of $N$ teachers and a set $S$ of $M$ schools. Each teacher $t \in T$ has a strict preference ordering $P_t$ over the set of schools and being unassigned, i.e., $S \cup \{\emptyset\}$. Let $R_t$ be the induced weak ordering. Each school $s \in S$ has a priority ordering $\succ_s$ over the teachers and leaving an empty position, i.e., $T \cup \{\emptyset\}$. We denote by $P$ the profile of preferences of the teachers, i.e., $P := (P_t)_{t \in T}$, and by $\succ$ the profile of priorities of the schools, i.e., $\succ := (\succ_s)_{s \in S}$. As usual, for a teacher $t \in T$, $P_t := (P_{t'})_{t' \neq t}$, and for a set $T' \subseteq T$, $P_{T'} := (P_{t'})_{t' \in T'}$. Each school $s$ has a capacity of $q_s$ seats, and we let $q := (q_s)_{s \in S}$ be the vector of capacities of the schools. A matching $\mu$ is a mapping from $T$ to $S \cup \{\emptyset\}$ where $\mu(t)$ is the school assigned to teacher $t$ where $\mu(t) = \emptyset$ means that teacher $t$ remains unassigned. We slightly abuse the notation in letting $\mu(s)$ be the set of teachers assigned to school $s$ and $\mu(\emptyset)$ be the set of unassigned teachers. A matching must respect the capacities of the schools, i.e., $\forall s \in S, |\mu(s)| \leq q_s$. The main difference between the college admission problem proposed by Gale and Shapley (1962) is that there is now an initial matching $\mu_0$. There are two types of teachers: i) newly tenured teachers who do not have any initial school, i.e., $\mu_0(t) = \emptyset$, and ii) tenured teachers who are initially assigned to a school, i.e., $\mu_0(t) \in S$. We let $n \leq N$ be the number of newly tenured teachers, i.e., $n := |\mu_0(\emptyset)|$. Note that if $n = N$, then we are back to a standard school choice model, and if $n = 0$, then the model becomes a housing market model where houses can have multiple seats and priorities. We also distinguish types of frameworks: i) a many-to-one setting if $\exists s \in S$ with $q_s > 1$ and ii) a one-to-one setting if $\forall s \in S, q_s = 1$. Moreover, we make two natural assumptions:

1. Every tenured teacher prefers his initial school to being unassigned, i.e., if $\mu_0(t) \in S$, then $\mu_0(t) P_t \emptyset$.
2. Every school finds all the teachers to be acceptable, i.e., $\forall t \in T$ and $s \in S$, $t \succ_s \emptyset$.

The first assumption is without loss of generality in our setting. Indeed, if a teacher $t$ prefers being unassigned to his/her initial school, we can consider him/her as being a newly tenured teacher, and all our results would hold. Concerning the second assumption, all our results would hold if one imposes the constraint that school $s$ cannot be assigned to teacher $t$ s.t. $\emptyset \succ_s t$ and that every school ranks its initial teachers as acceptable, i.e., if $t \in \mu_0(s)$, then $t \succ_s \emptyset$.

We say that a matching $\mu$ is:

- **Individually rational** (IR): $\forall t \in T$, $\mu(t) R_t \mu_0(t)$. An IR matching is s.t. all the teachers obtain an assignment that they weakly prefer to their initial one.
- **Stable**: $\exists(t, s) \in T \times S$ s.t. $s P_t \mu(t)$ and either i) $|\mu(s)| < q_s$ and $t \succ_s \emptyset$ or ii) $|\mu(s)| = q_s$ and $\exists t' \in \mu(s)$ s.t. $t' \succ_s t$. Such a pair is called a blocking pair. We also say, in the second case, that teacher $t$ has a justified envy toward teacher $t'$ at school $s$. With a slight abuse of language, we say that a matching with the existence of such pairs generates envy.\[9\]

\[9\] It is the ordering s.t. $s' R_t s \iff (s = s') \lor (s' P_t s)$.

\[10\] In the literature, the exact term is justified envy. For simplicity, we shorten the terminology without any risk of confusion.
- **Pareto-efficient (PE):** \( \exists \mu' \text{ s.t. } \forall t \in T, \mu'(t)R_t\mu(t) \) and the preference is strict for at least one teacher.

The set of PE and stable matchings can be disjoint so that any PE matching leads to potential blocking pairs. In a teacher assignment setting with some tenured teachers \((n < N)\), it is possible that no matching is both IR and stable such that any IR matching will have some remaining blocking pairs. For a given profile \((P, \succ)\) and matching \(\mu\), we let \(B_\mu(P, \succ)\) be the set of blocking pairs of \(\mu\) under the profile \((P, \succ)\), i.e., the set of pairs that respect the conditions in the above definition of stability. When the context is clear, we refer to \(B_\mu(P, \succ)\) as simply \(B_\mu\).

A mechanism \(\varphi\) maps each profile \((P, \succ)\) to a matching \(\varphi(P, \succ)\).\(^{11}\) A mechanism is:

- IR if \(\forall (P, \succ), \varphi(P, \succ)\) is an IR matching.
- Strategy-proof (SP) if \(\forall t \in T, \forall \succ, \forall P_t, P_t', P_t - t, \varphi_t(P_t, P_t - t, \succ) R_t \varphi_t(P_t', P_t - t, \succ)\).
- Stable if \(\forall (P, \succ), \varphi(P, \succ)\) is a stable matching.
- Pareto-efficient (PE) if \(\forall (P, \succ), \varphi(P, \succ)\) is a Pareto-efficient matching.

We consider concepts introduced by Abdulkadiroglu, Che, Pathak, Roth, and Tercieux (2017).\(^{12}\) We say that a matching \(\mu'\) has less envy (resp. strictly less envy) than a matching \(\mu\) if \(B_{\mu'} \subseteq B_{\mu}\) (resp. \(B_{\mu'} \subset B_{\mu}\)). A mechanism \(\varphi'\) has less envy than a mechanism \(\varphi\) if \(\forall (P, \succ), \varphi'(P, \succ)\) has less envy than \(\varphi(P, \succ)\). A mechanism \(\varphi'\) has strictly less envy than a mechanism \(\varphi\) if it has less envy than \(\varphi\) and \(\exists (P, \succ)\) s.t. \(\varphi'(P, \succ)\) has strictly less envy than \(\varphi(P, \succ)\). We say that \(\varphi\) is a minimal envy mechanism if there is no mechanism \(\varphi'\) that has strictly less envy than \(\varphi\).\(^{13}\) The next section will formally introduce the two mechanisms discussed in the introduction: DA* and TTC*.

### 3 Reassignment Mechanisms and Related Literature

#### 3.1 The modified Deferred Acceptance mechanism (DA*)

We start by recalling the definition of the well-known DA mechanism. In a school choice setting, it is the unique SP and stable mechanism.\(^{14}\)

**Step 1.** All the teachers apply to their first-ranked school. If a school \(s\) receives more than one application, it selects the best candidate according to \(\succ_s\) and rejects the others. If there is no rejection, then the algorithm stops. If there is at least one rejection, then move to Step 2.

**Step \(k \geq 2.\)** All the teachers rejected in step \(k - 1\) apply to their favorite school among those that have not rejected them yet. A school \(s\) considers both its previously accepted teachers, if any, and the candidates applying to it and selects the best according to \(\succ_s\). If there is no rejection, then the algorithm stops. If there is at least one rejection, then move to Step \(k + 1\).

If there are some tenured teachers \((n < N)\), because the DA mechanism is stable, it is not IR. Indeed, at any given step, it is possible for a school to reject its initially assigned teacher if another teacher with a higher priority applies. The following mechanism, first identified by Guillen and Kesten (2012) and later studied by Compte and Jehiel (2008) and Pereyra (2013), is a natural IR transformation of DA, which we call DA*:

---

\(^{11}\)Formally, a mechanism maps each tuple \((P, \succ, \mu_0)\) to a matching. Since we fixed the initial matching, we omit it here. In particular, an IR mechanism must always return an IR matching for any possible initial assignment.

\(^{12}\)Though different, it is also closely related to the relation “more stable than” introduced by Chen and Kesten (2017).

\(^{13}\)Formally, the relation “less envy than” defines an incomplete ordering \(\triangleright\) over a set \(X\) of mechanisms so that \((X, \triangleright)\) forms a partially ordered set (poset). In a poset \((X, \triangleright)\), \(x \in X\) is a maximal element if \(\nexists y \in X\) s.t. \(y \triangleright x\). When \(\triangleright\) is the “less envy than” relation, the minimal envy mechanisms in \(X\) are the maximal elements of \((X, \triangleright)\).

\(^{14}\)This is a standard result. We refer the reader to the survey of Abdulkadiroglu and Sonmez (2003) or the textbook of Roth and Sotomayor (1990).
1. Let $\succ$ be a modified priority profile s.t. $\forall s \in S$ and $\forall t, t' \in T$, if $t \in \mu_0(s)$ and $t' \notin \mu_0(s)$, then $t \succ_s t'$. If $\{t, t'\} \subseteq \mu_0(s)$ or $\{t, t'\} \cap \mu_0(s) = \emptyset$, then $t \succ_s t' \iff t \succ t'$. In words, $\succ_s$ moves all the initial teachers of school $s$ to the top of the priority ordering but keeps the relative rankings between them and the teachers not initially assigned to it.

2. Let $DA^*(P, \succ) = DA(P, \succ)$.

By construction, $DA^*$ is IR. Indeed, since any teacher is ranked first by his/her initial school, he/she would be sure to be accepted in the case he/she is applying to it. Since the modification of the priority profile does not depend on the reporting preferences of the teachers, it is also trivially SP. Compte and Jehiel (2008) and Pereyra (2013) showed that the matching returned by this mechanism respects a certain form of stability, called $\mu_0$-stable matching by Compte and Jehiel (2008) or acceptable matching by Pereyra (2013). A matching $\mu$ is acceptable if there is no teacher $t$, school $s$ and teacher $t' \in \mu(s) \setminus \mu_0(s)$ s.t. $SP_\mu(t)$ and $t \succ_s t'$. If such a situation occurs, Pereyra (2013) referred to it as a justified claim. Therefore, an acceptable matching is simply a matching with no justified claims. Since $DA^*$ returns an acceptable matching, if $SP_P DA_\mu(P, \succ)$ and $t \succ_s t'$ for $t' \in DA^*_P(P, \succ)$, then $\mu_0(t') = s$. This type of envy is called inappropriate claims by Pereyra (2013), and the only possible blocking pairs under $DA^*$ are formed by a teacher and school s.t. the latter is assigned to some of its initial teachers. Pereyra (2013) showed that there is no other acceptable matching that leads to a set of inappropriate claims included in one of $DA^*$ so that the latter minimizes inappropriate claims among acceptable matchings. Moreover, he showed that among all the acceptable mechanisms that minimize inappropriate claims, $DA^*$ is the most preferred by the teachers.

Our difference from this approach is that we do not differentiate between inappropriate or justified claims but consider the complete set of claims, i.e., all the blocking pairs, and ask whether it is possible to somehow “minimize it” in the setwise inclusion sense while maintaining strategy-proofness. This process is motivated by the observation that for some profile $(P, \succ)$, it is possible to find a matching $\mu$ that has strictly less envy than $DA^*(P, \succ)$, i.e., $B_\mu \subset B_{DA^*(P, \succ)}$. In particular, Combe, Tercieux, and Terrier (2016) proposed an IR and strategy-proof mechanism, the TO-BE mechanism, that selects such matching for some profile. However, it is not comparable with $DA^*$ according to the “less envy than” ordering. Therefore, the question of whether one can find an IR and SP mechanism that has (strictly) less envy than $DA^*$, i.e., whether the latter is a minimal envy mechanism, is open, and our Proposition 1 will show that it is indeed not possible.

In a school choice setting, Abdulkadiroglu, Pathak, and Roth (2005) showed that there is no mechanism that is SP and always leads to a matching that weakly (and sometimes strictly) Pareto-dominates that of DA. The Pareto-dominance ordering between two mechanisms, similarly to the “less envy than” ordering, defines an incomplete ordering over a set of mechanisms. Similarly to minimal envy mechanisms, one can study the “maximally efficient” mechanisms using this relation. That is exactly what the result of Abdulkadiroglu, Pathak, and Roth (2005) shows for DA in a school choice setting: in the set of SP mechanisms, DA is a “maximally efficient” mechanism. Recently, Kesten and Kurino (2017) extended this result to the case where students are somehow restricted in their preference profiles. They provided the maximal domain of preferences under which it is possible to find an SP mechanism. When the manipulations are restricted to this domain, Pareto dominates DA. Even if the question is mathematically similar, i.e., finding maximal elements of a partially ordered set of mechanisms, the study of minimal envy mechanisms is conceptually different. Indeed, if between any two matchings $\mu$ and $\mu'$, $B_{\mu'} \subseteq B_{\mu}$, then one cannot conclude that $\mu'$ Pareto dominates $\mu$, so the aforementioned results cannot be used. In particular,

\footnote{However, note that Pereyra (2013)’s result implies that $\mu$ will turn an inappropriate claim into a justified one.}

\footnote{We refer the reader to their paper for the technical details. Formally, some students do not have access to the possibility of remaining unassigned ($\emptyset$), so the proof in Abdulkadiroglu, Pathak, and Roth (2005), which crucially uses such possibility, cannot be applied.}
one can easily construct examples where teachers can be worse off under \( \mu' \).\(^{17}\) However, we show that the analysis of the DA mechanism done in this literature are useful for our question. Indeed, the use of *weakly underdemanded schools* under DA\(^*\), which are schools that all the teachers prefer weakly less than their assignment under DA\(^*\), proved to be key for our result. We borrow one technical lemma from Kesten and Kurino (2017) (Lemma 1) concerning these schools to prove our Proposition 1.

### 3.2 The generalized Top Trading Cycle mechanism (TTC\(^*\))

As before, we start by recalling the definition of the TTC mechanism introduced by Abdulkadiroglu and Sonmez (2003) in the school choice setting. It is an SP and PE mechanism. Since our result will hold only in a one-to-one framework, we consider only a one-to-one setting when defining the mechanism, so \( \forall s \in S, q_s = 1 \).

**Step 1.** Build the graph where the nodes are the teachers and schools. Every teacher points to his/her favorite school, if he/she prefers staying unassigned, let him/her point to him/herself. Every school points to its highest-ranked teacher. There will be a cycle in this graph, and all cycles will involve different nodes,\(^{18}\) implement one in letting the teachers be assigned to the school they are pointing to. Delete these teachers and their assigned schools.\(^{19}\)

**Step \( k \geq 2 \).** Build the graph where the nodes are the remaining teachers and the remaining schools. Repeat the same operation as in Step 1 to implement a cycle in this graph.

The process continues until all the teachers obtain an assignment. As before, if there are some tenured teachers, the above mechanism is not IR. If a school does not rank its initial teacher first, it can point to another teacher and be assigned before its initial teacher such that it does not ensure the latter of obtaining a weakly preferred school compared to his/her initial one. Similarly to the construction of DA\(^*\), one can apply the same technique to define an IR version of the above mechanism, which we call TTC\(^*\):

1. Let \( \succ \) be a modified priority profile s.t. \( \forall s \in S \) and \( \forall t, t' \in T \), if \( t \in \mu_0(s) \) and \( t' \notin \mu_0(s) \), then \( t \succeq_s t' \). If \( \{t, t'\} \subseteq \mu_0(s) \) or \( \{t, t'\} \cap \mu_0(s) = \emptyset \), then \( t \succeq_s t' \iff t \succeq t' \). In words, \( \succ \) moves all the initial teachers of school \( s \) to the top of the priority ordering but keeps the relative rankings them and the teachers not initially assigned to it.

2. Let \( \text{TTC}^*(P, \succ) = \text{TTC}(P, \succ) \).

It can be easily verified that the above mechanism is IR and SP. Additionally, since we only modified the priorities of the schools, the TTC\(^*\) mechanism is also Pareto-efficient.

In a housing market setting with only tenured teachers \( (n = 0) \), this mechanism is equivalent to Gale’s Top Trading Cycle algorithm, so Ma (1994) showed that it is the only IR, SP and PE mechanism. In a school choice setting \( (n = N) \), it becomes equivalent to the above school choice TTC mechanism, and Abdulkadiroglu, Che, Pathak, Roth, and Tercieux (2017) showed that it is a minimal envy mechanism in the set of SP and PE mechanisms. In an intermediate model \( (0 < n < N) \), this mechanism is equivalent to the *top trading rule* introduced by Karakaya, Klaus, and Schlegel (2017).\(^{20}\) Moreover, if schools all have

\(^{17}\) In a school choice setting, simply compare the student-optimal stable matching to the school-optimal stable matching. Indeed, the two trivially lead to the same set of blocking pairs, the empty set, and the former Pareto dominates the latter.

\(^{18}\) A cycle is a set of nodes \( \{t_1, s_1, \ldots, t_K, s_K\} \) such that for \( k = 1, \ldots, K - 1, t_k \) points to \( s_k \), \( s_k \) points to \( t_{k+1} \), and \( s_K \) points to \( t_1 \). The existence of a cycle and the fact that they are disjoint is a standard result for graphs where nodes have only one outgoing edge. The order in which the cycles are implemented does not influence the outcome of the algorithm.

\(^{19}\) In a many-to-one setting, one would keep the schools that have additional seats available.

\(^{20}\) Their motivation is different than ours since they consider a housing market model and we consider a school choice one. They characterize all the mechanisms that are SP, PE, consistent and reallocation-proof and show that any such mechanism is equivalent to a top trading rule for some well-constructed priority profile. In our setting, the priority profile is part of the primitives and fixed, and we focus on the envy that such mechanism generates.
the same priorities, i.e., $\forall s,s' \in S$, $s \succ s'$, then TTC$^*$ is equivalent to the YRMH-IGYT mechanism introduced by Abdulkadiroglu and Sonmez (1999) and later characterized by Sonmez and Unver (2010). The question of whether this mechanism is, in a one-to-one setting, a minimal envy mechanism in the set of IR, SP and PE mechanisms in an intermediate setting with $0 < n < N$, is open, and our Proposition 3 confirms that it is indeed the case.

4 Results

In a school choice setting ($n = N$), if one considers the set of SP mechanisms, then DA has trivially less envy than TTC. In our setting, with some initially assigned teachers ($n < N$), we start by showing that DA$^*$ and TTC$^*$ cannot be compared in terms of the less envy criterion.

**Example 1.** There are 3 teachers, $t_1$, $t_2$, and $t_3$, all initially assigned to, respectively, $s_1$, $s_2$, and $s_3$. The preferences and priorities are given by:

- $P_{t_1} : s_2 \succ s_3 \succ s_1 \succ_{s_1} t_3 \succ t_2 \succ t_1$
- $P_{t_2} : s_1 \succ s_2 \succ s_3 \succ_{s_2} t_3 \succ t_1 \succ t_2$
- $P_{t_3} : s_1 \succ s_2 \succ s_3 \succ_{s_3} t_1 \succ t_3 \succ t_2$

Under the above profile $(P,\succ)$, we have:

- $DA^*(P,\succ) = \begin{pmatrix} t_1 & t_2 & t_3 \\ s_3 & s_2 & s_1 \end{pmatrix}$
- $TTC^*(P,\succ) = \begin{pmatrix} t_1 & t_2 & t_3 \\ s_2 & s_1 & s_3 \end{pmatrix}$

Note that $B_{DA^*(P,\succ)} = \{(t_1,s_2)\}$ and $B_{TTC^*(P,\succ)} = \{(t_3,s_1),(t_3,s_2)\}$, so the two sets do not intersect.\(^{21}\)

The next proposition shows that, in the set of IR and SP mechanisms, DA$^*$ is a minimal envy mechanism.

**Proposition 1.** DA$^*$ is a minimal envy mechanism in the set of IR and SP mechanisms.

The proof shows that there is no IR and SP mechanism that has strictly less envy than DA$^*$. The structure of the matchings that have less envy than that of DA$^*$ must be understood, so several lemmas are dedicated to this study. The lemmas provide interesting results, so we relegate their formal treatment to Section 5. To provide intuition about the proof and its related lemmas, consider the following simple example: there are five teachers $t_1, \ldots, t_5$ and five schools $s_1, \ldots, s_5$ with one seat each. Teacher $t_k$, for $k = 1, 2, 3, 4$, is a tenured teacher and is initially assigned to school $s_k$. Teacher $t_5$ is a newly tenured teacher with no initial assignment, so school $s_5$ is initially vacant. Preferences and priorities are given by:

- $P_{t_1} : s_2 \succ s_1 \succ s_3 \succ_{s_1} t_3 \succ t_2 \succ t_1 \ldots$
- $P_{t_2} : s_1 \succ s_2 \succ \ldots \succ_{s_2} t_3 \succ t_1 \succ t_2 \ldots$
- $P_{t_3} : s_1 \succ s_2 \succ s_3 \succ_{s_3} t_3 \succ t_1 \succ t_2 \ldots$
- $P_{t_4} : s_5 \succ s_4 \succ \ldots \succ_{s_4} t_4 \succ t_5 \ldots$
- $P_{t_5} : s_4 \succ s_5 \ldots \succ_{s_5} t_5 \succ t_4 \ldots$

Dots indicate that preferences and priorities can be arbitrary. Let $(P,\succ)$ be the above profile of preferences and priorities. Under this profile, one can check that the matching given by DA$^*$ is:

$$DA^*(P,\succ) = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 \\ t_1 & t_2 & t_3 & t_5 & t_4 \end{pmatrix}$$

\(^{21}\)It is possible to construct examples where the set of blocking pairs of TTC is strictly included in that of DA$^*$ and others where the reverse occurs.
Note that the set of blocking pairs is \( B_{DA^* (P, >)} = \{(t_1, s_2), (t_2, s_1), (t_3, s_1), (t_3, s_2)\} \). Now, consider the following matching:

\[
\mu' = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 \\ t_2 & t_1 & t_3 & t_4 & t_5 \end{pmatrix}
\]

To go from \( DA^* (P, >) \) to \( \mu' \), \( t_1 \) and \( t_2 \) exchange their assignments, as do \( t_4 \) and \( t_5 \). Lemma 2 in Section 5 shows that going from a matching \( \mu \) to a matching \( \mu' \) can be decomposed into exchanges of seats through cycles and possibly chains if some vacant seats exist or if some agents are unmatched. Note that \( B_{\mu'} = \{(t_3, s_1), (t_3, s_2)\} \subset B_{DA^* (P, >)} \), so \( \mu' \) has strictly less envy than \( DA^* (P, >) \) but \( \mu' \) does not Pareto dominate \( DA^* (P, >) \): \( t_4 \) and \( t_5 \) are strictly worse off. However, we note a particular structure: teachers who exchange their assignments from \( DA^* \) to \( \mu' \) are either all strictly better off or all strictly worse off. Indeed, teachers \( t_1 \) and \( t_2 \) exchange their assignments and are both strictly better off under \( \mu' \) than under \( DA^* (P, >) \). Teachers \( t_4 \) and \( t_5 \) also exchange their assignments and are both strictly worse off. Lemma 3 in Section 5 shows that this is a general property: when considering a matching with less envy than \( DA^* \), all teachers involved in an exchange must either all be strictly better off or all strictly worse off. Note that in our example, \( B_{\mu'} \) is a strict subset of \( B_{DA^* (P, >)} \) and that at least one exchange involves teachers who are all strictly better off. Lemma 4 shows that it is generally the case that if a matching has strictly less envy than that of \( DA^* \), at least one exchange must make all its teachers strictly better-off.\(^{22}\)

Assume that a mechanism \( \varphi \) has strictly less envy than \( DA^* \) and that under the profile \( (P, >) \) of our example, \( \varphi (P, >) = \mu' \) so \( B_{\varphi (P, >)} \subset B_{DA^* (P, >)} \). To provide a full intuition for Proposition 1, we will now use the terminologies and one technical lemma of Kesten and Kurino (2017). Consider the exchange between \( t_1 \) and \( t_2 \). We have \( \varphi_{t_1} (P, >) = s_2 P_t_1 s_1 = DA^*_{t_1} (P, >) \). We also know that all the teachers in this exchange are strictly better off, so \( \varphi_{t_2} (P, >) = s_1 P_t_2 s_2 = DA^*_{t_2} (P, >) \). Since \( t_2 \) strictly prefers \( s_1 \) to his/her assignment under \( DA^* (P, >) \), \( s_1 \) is called an overdemanded school under \( DA^* (P, >) \) at the preference profile \( P \). Therefore, under \( DA^* (P, >) \), \( t_1 \) is assigned to \( s_1 \), an overdemanded school. Now, note that under the matching \( DA^* (P, >) \) and profile \( P \), \( s_3 \) is a school s.t. all teachers weakly prefer their assignment to \( s_3 \), i.e., for \( k = 1, 2, 3, 4, 5 \), \( DA^*_{t_k} (P, >) R_{t_k} s_3 \).\(^{23}\) Such school is called a weakly underdemanded school.

Consider \( t_1 \) reporting profile \( P'_{t_1} \) that is the same as \( P_t_1 \), except that it upgrades \( s_3 \) right above \( s_1 \), where \( s_1 \) is \( t_1 \)'s assignment that is overdemanded under \( DA^* (P, >) \) at \( P \):

\[
P'_{t_1} : s_2 \ s_3 \ s_1
\]

With the new profile \( P' := (P'_{t_1}, P_{-t_1}) \), we have:

\[
DA^* (P', >) = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 \\ t_3 & t_2 & t_1 & t_5 & t_4 \end{pmatrix}
\]

Therefore, \( t_1 \) is assigned to \( s_3 \) under \( DA^* \). Furthermore, note that \( s_3 \) is still underdemanded under \( DA^* (P', >) \) at the new profile \( P' \). Kesten and Kurino (2017)'s Lemma 1 shows that in a school choice setting, a teacher assigned to an overdemanded school under \( DA \) can always upgrade an underdemanded school immediately above its assigned school and be assigned to the former while such a school remains weakly underdemanded at the new preference profile. This result can easily be extended to our teacher assignment setting, as stated in Lemma 6. Now, in our example, under the new profile \( (P', >) \), we have \( B_{DA^* (P', >)} := \{(t_1, s_2)\} \). Since the mechanism \( \varphi \), has strictly less envy than \( DA^* \), then, at the profile \( (P', >) \), it selects a matching \( \tilde{\mu} \) s.t. \( B_{\tilde{\mu}} \subset B_{DA^* (P', >)} \). Note that, under the profile \( (P', >) \), there are only 3 other possible matchings between \( t_1, t_2 \) and \( t_3 \) compared to \( DA^* (P', >) \): 1) \( t_1 \) is assigned to \( s_2 \), \( t_2 \) to \( s_1 \) and \( t_3 \) stays at \( s_3 \), 2) \( t_1 \) is assigned to \( s_3 \), \( t_3 \) to \( s_2 \) and \( t_2 \) to \( s_1 \) or 3) the three of them stay at their

\(^{22}\)Similarly to the remark in Footnote 17, it is possible to find two different matchings with the same set of blocking pairs but where one is Pareto dominated by the other. In our example, simply consider the matching that exchanges the assignments of \( t_4 \) and \( t_5 \).

\(^{23}\)Note that the relation requires a weak preference so that it holds for \( t_3 \) for whom \( DA^*_{t_3} (P, >) = s_3 \).
initial school. The three resulting matchings all have, at least, \((t_3, s_3)\) as a blocking pair so that they do not generate less envy than \(DA^*(P', \succ)\). So necessarily, \(\varphi_{t_1}(P', \succ) = DA^*_{t_1}(P', \succ)\) so that, trivially, \(DA^*_t(P', \succ) = s_3P_{t_1}t_{t_1}(P', \succ)\). Lemma 7 shows that the latter relation holds generally, so any teacher assigned to a weakly underdemanded school under \(DA^*\) must be assigned to a weakly less preferred school under any mechanism with less envy than \(DA^*\). In our example, since \(s_2P'_{t_1}s_3\), we have:

\[
\varphi_{t_1}(P, \succ) = s_2P'_{t_1}s_3 = DA^*_{t_1}(P', \succ)P'_{t_1}\varphi_{t_1}(P', \succ)
\]

Therefore, \(\varphi_{t_1}(P, \succ)P'_{t_1}\varphi_{t_1}(P', \succ)\) and since \(P\) and \(P'\) only differ because of the preference profile reported by \(t_1\), we conclude that \(\varphi\) is not strategy-proof. In Section 5, we formally prove that the above intuitions hold generally.

Proposition 1 shows that no IR and SP mechanism has strictly less envy than \(DA^*\). One can ask whether it is possible to find an IR and SP mechanism \(\varphi\) different from \(DA^*\) and that generates the same envy, i.e. \(\forall(P, \succ), B_{\varphi(P, \succ)} = B_{\varphi(P, \succ)}\). Example 2 shows that it is indeed possible.

**Example 2.** Consider a setting with 3 teachers, \(t_1, t_2,\) and \(t_3\), who are initially assigned to, respectively, \(s_1, s_2,\) and \(s_3\), each having only one seat. Let \(\varphi\) be the following mechanism. If the priority profile \(\succ\) is:

\[
\succ s_1 t_1 t_3 t_2 \\
\succ s_2 t_2 t_3 t_1 \\
\succ s_3 t_3 t_1 t_2
\]

Then \(\forall P\), let \(\varphi(P, \succ) = \mu_0\), i.e. the trivial mechanism where no teacher moves from his/her initial school. Otherwise, if the priority profile or the number of teachers differ, let \(\varphi(P, \succ) = DA^*(P, \succ)\).

It is clear that \(\varphi\) is an IR mechanism. It is also trivially strategy-proof. Indeed, teachers cannot influence the priority profile of the schools and both the trivial mechanism, if the priority profile is the above one, and \(DA^*\) otherwise, are SP mechanisms. Last, note that, when the priority profile is the above one, since all the schools rank their initial teacher first, then in the Step 1 of \(DA^*\), there is no modification of the priority profile so \(DA^*(P, \succ) = DA(P, \succ)\) and the resulting matching is stable at any preference profile \(P\), i.e. \(B_{DA^*(P, \succ)} = \emptyset\). Since \(\varphi(P, \succ) = \mu_0\) for this priority profile, then each school is assigned its first ranked teacher so that \(\varphi(P, \succ) = \emptyset = B_{DA^*(P, \succ)}\). Since when the priority profile differs from the above one, we have \(\varphi(P, \succ) = DA^*(P, \succ)\), then trivially \(B_{\varphi(P, \succ)} = B_{DA^*(P, \succ)}\) and we conclude that \(\varphi\) and \(DA^*\) generate the same envy even though they differ from each other.

Example 2 highlights an important contrast with the school choice literature (\(n = N\)) since Alcalde and Barberà (1994) (Theorem 3) showed that, in a school choice framework, DA is the unique SP and stable mechanism. If there are tenured teachers (\(n < N\)), we assumed that they all find their initial school acceptable so that this restriction on the preference domain, in addition to the IR constraint, imply that these teachers can never end up being unassigned under any IR mechanism. This allows us to exhibit additional mechanisms that are IR, SP and stable under the modified priority profiles that rank the initial teachers at the top of their initial schools.

We now turn to the analysis of \(TTC^*\). We start by showing that \(TTC^*\) is not a minimal envy mechanism in the set of IR and SP mechanisms:

---

24 However, it is possible to find an example where the preference relation is strict so that the teacher would strictly prefer his/her assignment under \(DA^*\) than under the other mechanism. In our example, \(DA^*_t(P, \succ) = s_4\) and \(s_4\) is also weakly underdemanded under \(DA^*(P, \succ)\) at \(P\) but \(s_4P_{t_4}t_{t_4}(P, \succ) = s_5\).

25 Example 2 can also be used to show that, in a school choice setting, if one considers the preference domain where all students find all schools acceptable; then one can exhibit stable mechanisms that are SP in this restricted preference domain and that differ from the DA algorithm. The argument of the proof of Theorem 3 in Alcalde and Barberà (1994) crucially relies on the possibility that any student can remain unassigned. Moreover, Example 2 implies that the Corollary 3 of Guillen and Kesten (2012), that states that \(DA^*\) is the unique IR, SP and fair (in the sense of \(\mu_0\)-stable or acceptable as defined in Section 3) mechanism, is wrong.
Proposition 2. There are mechanisms that are IR, SP and have strictly less envy than TTC*.

Proof. Consider a setting with 3 teachers, t₁, t₂, and t₃, who are initially assigned to, respectively, s₁, s₂, and s₃, each having only one seat. Now, let ϕ be the following mechanism: if the priority profile ⊳ is

\[
\begin{align*}
    &\succ_{s_1} t_1 \ t_3 \ t_2 \\
    &\succ_{s_2} t_2 \ t_3 \ t_1 \\
    &\succ_{s_3} t_3 \ t_1 \ t_2 
\end{align*}
\]

Then, let ϕ(P, ⊳) = DA*(P, ⊳). Otherwise, if ⊳ differs from the above profile or if the number of teachers differs, let ϕ(P, ⊳) = TTC*(P, ⊳). This mechanism is trivially IR, and it is easy to see that it is also SP because the teachers cannot influence the priority profile of the schools and for a fixed profile, the mechanism used is an SP mechanism. Since ϕ = TTC*, when ⊳ differs from the above profile or when the number of teachers differs, then the two mechanisms have trivially the same set of blocking pairs. Note that when the priority profile is the above profile, since all the schools rank their initial teachers first, then in the Step 1 of DA*, there is no modification of the priority profile so DA*(P, ⊳) = DA(P, ⊳) and the resulting matching is stable at any preference profile P. Therefore, Bϕ(P, ⊳) = ∅ and is trivially included in B_{TTC*}(P, ⊳). However, note that when t₁ ranks s₂ first and t₂ ranks s₁ first, then:

\[
\text{TTC}^*(P, ⊳) = \left( \begin{array}{ccc} t_1 & t_2 & t_3 \\ s_2 & s_1 & s_3 \end{array} \right)
\]

And Bϕ(P, ⊳) = ∅ ⊂ B_{TTC*}(P, ⊳) = \{(t₃, s₁), (t₃, s₂)\}, so that ϕ has indeed strictly less envy than TTC*. □

At that point, one can wonder whether DA* is the only minimal envy mechanism in the set of IR and SP mechanisms. Since the “less envy than” ordering is incomplete, one can easily show that it is indeed not the case. Suppose that when there are 3 teachers and n = 0, as in Example 1, one uses the TTC* mechanism and if the number of teachers differs, uses DA*. When the profile (P, ⊳) is the same as in Example 1, we have seen that B_{DA*}(P, ⊳) = \{(t₁, s₂)\} and B_{TTC*}(P, ⊳) = \{(t₃, s₁), (t₃, s₂)\}. As we will see, TTC* is not a minimal envy mechanism in the set of IR and SP mechanisms. Let ϕ be a minimal envy, IR and SP mechanism with strictly less envy than TTC*. Note that in Example 1, at profile (P, ⊳), there is no IR and stable matching, i.e., there is no IR matching μ s.t. μₚ = ∅. Specifically, ∅ ≠ Bₚ(ϕ(P, ⊳)) ⊂ B_{TTC*}(P, ⊳) = \{(t₃, s₁), (t₃, s₂)\} and so ϕ and DA* cannot be compared in terms of blocking pairs.

This negative result for TTC* parallels the standard result in a school choice setting concerning the standard TTC: if one considers only the set of strategy-proof mechanisms, then TTC is not a minimal envy mechanism in that set since, trivially, the DA algorithm always leads to less envy, i.e., no blocking pairs. However, as mentioned, Abdulkadiroglu, Che, Pathak, Roth, and Tercieux (2017) showed that the TTC algorithm in a one-to-one setting is a minimal envy mechanism in the set of SP and PE mechanisms. Our next proposition extends their result to our teacher assignment setting:

Proposition 3. In a one-to-one setting, TTC* is a minimal envy mechanism in the set of IR, SP and PE mechanisms.

The proof of the proposition extends that of Abdulkadiroglu, Che, Pathak, Roth, and Tercieux (2017) and is relegated to Section A of the Appendix. The proof shows that any IR, SP and PE mechanism ϕ with (weakly) less envy than TTC* must be equal to it. This is achieved by induction over the steps of TTC*. For instance, in the first step of TTC*(P, ⊳), suppose that teacher t was assigned to school s’ and that school s was pointing to t during cycle C that assigned him/her at that step. If ϕ(P, ⊳) assigns t to a school s'' ≠ s’, then s’Pₜs'' since t was pointing to his/her favorite school at the first step of TTC* and s'' was still present at that step. In a school choice setting (n = N), if t reports profile Pₜ : s’, s, ∅, then the matching of TTC* remains the same and Abdulkadiroglu, Che, Pathak, Roth, and Tercieux (2017) showed that ϕₜ(Pₜ, P₋ₜ, ⊳) = s. One can then iterate the argument to reach a profile Pₜ where
TTC\(^{*}(P', \succ) = \text{TTC}\(^{*}(P, \succ)\) and all the teachers in cycle \(C\) are assigned under \(\varphi(P', \succ)\) to the school that was pointing to them under \(C\). This case contradicts that \(\varphi\) is PE since we know that, because of the cycle \(C\), all these teachers would be better off in exchanging their assignments. In the case where there are tenured teachers \((n < N)\), our proof uses a similar argument, except now, one has to consider whether school \(s\) is the initial school of teacher \(t\). If not, a similar argument as that in the school choice setting applies. If \(\mu_0(t) = s\), then \(t\) is not necessarily the highest-ranked teacher in the priority profile of \(s\), but one can use the fact that \(\varphi\) is IR to show that \(\varphi_\ell(P'_t, P_{-t}, \succ) = s\). The proof shows that this distinction leads to a conclusion similar to that in the school choice setting.

Similarly to the result of Abdulkadirouglu, Che, Pathak, Roth, and Tercieux (2017), the one-to-one assumption is necessary for Proposition 3 to hold. In a many-to-one setting, using similar arguments, one can exhibit an IR, SP and PE mechanism that has strictly less envy than TTC\(^{*}\). One can also consider whether TTC\(^{*}\) is the unique minimal envy mechanism in the set of IR, SP and PE mechanisms. Remember that when there are no newly tenured teachers \((n = 0)\), the model becomes a housing market problem and TTC\(^{*}\) becomes the TTC mechanism. We know by Ma (1994) that the latter mechanism is the unique IR, SP and PE mechanism. When there are only newly tenured teachers, i.e., \(n = N\), Abdulkadirouglu, Che, Pathak, Roth, and Tercieux (2017) provide an example of an SP, PE and minimal envy mechanism that is different from the school choice TTC mechanism. Since in that setting, TTC\(^{*}\) becomes equivalent to the latter, it is easy to see that their example can be extended to our setting to show that TTC\(^{*}\) is not the only minimal envy mechanism in the set of IR, SP and PE mechanisms.

5 Proof of Proposition 1

Before moving to the proof, we elaborate several lemmas that help to understand the structure of matchings that lead to less envy than DA\(^{*}\). First, we recall an obvious property of DA\(^{*}\) that is a direct consequence of Pereyra (2013). It simply states that if a teacher is blocking with a school under the matching of DA\(^{*}\), then he can only have a higher priority than some initial teachers of that school who remained assigned to it.

**Lemma 1.** For a profile \((P, \succ)\), if \((t, s) \in B_{DA^\ast}(P, \succ)\), then \(|DA^\ast_s(P, \succ)| = q_s\) and for all \(t' \in DA^\ast_s(P, \succ)\) s.t. \(t' \succ t\), \(t' \in \mu_0(s)\).

The following lemmas give some properties of a matching leading to a set of a blocking pairs that is a subset of that of DA\(^{*}\). We start with the following lemma that states that one can decompose the move from one IR matching to another as cycles and chains in a well-defined graph:

**Lemma 2.** Take any two IR matchings \(\mu\) and \(\mu'\). Build the graph \((N, E)\) where\(^{26}\)

\[
N := \left( \bigcup_{t \in T} \{ (t, \mu(t)) \} \right) \cup \left( \bigcup_{s: |\mu(s)| < q_s} \bigcup_{k=1}^{q_s-|\mu(s)|} \{ (\emptyset_k, s) \} \right) \cup \{ (\emptyset, \emptyset) \} := N_1 \cup N_2 \cup \{ (\emptyset, \emptyset) \}
\]

Let \(E\) be the edges s.t. i) \((t, s) \in T \times (S \cup \{\emptyset\})\) points to \((t', s')\) with \(s' \in S\) iff \(s' = \mu'(s)\) (with possible self-pointing if \((t, s) = (t', s')\) and \(t, s \in T \times (S \cup \{\emptyset\})\) points to \((\emptyset, \emptyset)\) iff \(\mu'(t) = \emptyset\).\(^{27}\) Then, there exists a collection of subsets \(C_1, \ldots, C_K\) of \(N\) \(\{ (\emptyset, \emptyset) \} \) s.t. \(\forall k = 1, \ldots, K, C_k := \{ (t^k_1, s^k_1), \ldots, (t^k_{n_k}, s^k_{n_k}) \}\), and i) these subsets form a partition of \(N\) \(\{ (\emptyset, \emptyset) \} \), ii) \(\forall \ell = 1, \ldots, n_k - 1, s^k_{\ell+1} = \mu'(t^k_{\ell}) \neq \emptyset\), i.e., \([(t^k_\ell, s^k_\ell), (t^k_{\ell+1}, s^k_{\ell+1})] \in E\), and either:

1. \(t^k_{n_k} \in T\) and \(s^k_1 = \mu'(t^k_{n_k}) \in S\) so that \([ (t^k_{n_k}, s^k_{n_k}), (t_1, s_1) ] \in E\). In that case, \(C_k\) is called a cycle.

\(^{26}\)The nodes in \(N_1\) represent teachers and their assignment and nodes in \(N_2\) represent empty seats in schools that do not fill their capacity under \(\mu\). The last node \((\emptyset, \emptyset)\) is used for teachers who are assigned under \(\mu\) but not under \(\mu'\).

\(^{27}\)Note that, in this case, the IR condition implies that \(t\) is a newly tenured teacher, i.e., \(\mu_0(t) = \emptyset\).
2. Or \((t_{nk}, s_{nk}) \in N_2 \cup \{(\emptyset, \emptyset)\}\). In that case, \(C_k\) is called a chain.

This collection represents cycles and chains of nodes in the graph \((N, E)\). Cycles respect condition (1) and chains respect condition (2). We refer to these subsets as the decomposition from \(\mu\) to \(\mu'\).\(^{28}\)

Proof. Let \(\tilde{T} := \{t \in T : \mu(t) \neq \mu'(t)\}\) and let \(\tilde{n} := |\tilde{T}|\).

If a teacher \(t\) does not change his/her allocation from \(\mu\) to \(\mu'\), i.e., \(t \notin \tilde{T}\), then \(\mu(t) = \mu'(t) := s\) (with possible \(s = \emptyset\)) and trivially, in the graph defined in the lemma, the node \((t, s)\) points to itself. Let \(C_1 := \{(t, s)\}\) such that this set trivially respects condition (1) of the lemma. There are \(N - \tilde{n}\) such teachers, so one can create \(C_1, \ldots, C_{N - \tilde{n}}\) singleton sets representing cycles of teachers who do not move between \(\mu\) and \(\mu'\).

Now, consider the subgraph \((\tilde{N}_1, \tilde{E}_1)\), where \(\tilde{N}_1\) deletes any nodes used in the above step and \(\tilde{E}_1\) deletes any edge ending at one of the nodes of an above teacher, i.e., \(\tilde{N}_1 = N \setminus \bigcup_{k=1}^{N - \tilde{n}} C_k\) and \(\tilde{E}_1 := E \setminus \{(t, s), (t', s') \in E : t' \notin \tilde{T}\}\). Build a path of nodes in \((\tilde{N}_1, \tilde{E}_1)\) as follows: fix a teacher \(t \in \tilde{T}\) and in the new subgraph with edges in \(\tilde{E}_1\), select any of its outgoing edges starting at node \((t, s) \in T \times (S \cup \{\emptyset\})\) and continue the same process for the node it is pointing to. Note that by condition (1) of the pointing, and finiteness of the environment, this path must either:

1. Cycle to the node of some teacher \(\tilde{t}\) of the path. In that case, this cycle only involves nodes of the form \((t, s) \in T \times S\).
2. End with a node without any outgoing edge, i.e., either the empty node \((\emptyset, \emptyset)\) or a node in \(N_2\). In that case, one can complete the chain by following the edges backwards if there is any node pointing to \((t, s)\). Note that since by construction of the pointing, all nodes have an outdegree of at most one, then this procedure cannot cycle and must end with a node of the form \((t', s') \in T \times (S \cup \{\emptyset\})\) with no ingoing edge.

One can easily check that with an appropriate labeling, these two cases identify a set \(C_{N - \tilde{n} + 1}\) representing either a cycle and chain in the statement.

Now consider the subgraph \((\tilde{N}_2, \tilde{E}_2)\) s.t. \(\tilde{N}_2 := \tilde{N}_1 \setminus C_{N - \tilde{n} + 1}\) and \(\tilde{E}_2 := \tilde{E}_1 \setminus \{(n, n') \in \tilde{E}_1 : n' \in C_{N - \tilde{n} + 1}\}\). One can again follow the same procedure as that of the subgraph \((\tilde{N}_1, \tilde{E}_1)\) to pin down a set of nodes \(C_{N - \tilde{n} + 2}\) that represents a cycle or a chain.

Continuing the argument, since at each step one defines a strict subgraph, this procedure will end and will define sets of nodes that, by construction, respect the conditions of the lemma. \(\square\)

The following lemma exhibits the property of decomposition from the matching \(DA^*(P, >)\) to a matching \(\mu\) with \(B_\mu \subseteq B_{DA^*(P, >)}\). It states that each cycle or chain in the decomposition from \(DA^*(P, >)\) to \(\mu\) that involves more than one teacher must either make all its teachers strictly better off or strictly worse off.

**Lemma 3.** Fix an IR matching \(\mu\) s.t. \(B_\mu \subseteq B_{DA^*(P, >)}\) and a decomposition \(C_1, \ldots, C_K\) as defined in Lemma 2 from \(DA^*(P, >)\) to \(\mu\). Fix any \(k = 1, \ldots, K\) s.t. \(n_k > 1\), then either:

1. \(\forall \ell = 1, \ldots, n_k: \mu(t_k^\ell) = s_{k+1, \ell} P^k t_k^\ell s_k^\ell = DA^*_{t_k^\ell}(P, >)\).
2. \(\text{Or } \forall \ell = 1, \ldots, n_k: DA^*_{t_k^\ell}(P, >) = s_k^\ell P^k t_k^\ell s_{k+1, \ell} = \mu(t_k^\ell)\).

Proof. Fix a set \(C_k := \{(t_1, s_1), \ldots, (t_{n_k}, s_{n_k})\}\) with \(n_k > 1\). W.l.o.g. fix teacher \(t_1\) and assume that \(\mu(t_1) = s_2 P_1 DA^*_{t_1}(P, >)\) so that, by IR, \(s_2 \neq \emptyset\). First, note that we cannot have \(|DA^*_{s_2}(P, >)| < q_{s_2}\).\(^{28}\)

\(^{28}\)In a many-to-one framework, this decomposition is not unique, but our results apply to any such decomposition.
Indeed, if this was the case, then $t_1$ would block the matching of $DA^*(P,\succ)$ with the school not filling all its seats, a contradiction to Lemma 1. Therefore, $|DA^*_{s_2}(P,\succ)| = q_{s_2}$ and $t_2 \neq \emptyset$.

Case 1. $t_2 \in DA^*_{s_2}(P,\succ) \setminus \mu_0(s_2)$.

First note that trivially, since $t_2$ is assigned to $s_2$ under $DA^*(P,\succ)$, we have that $(t_2, s_2) \notin B_{DA^*(P,\succ)}$. Using Lemma 1, since $t_2 \in DA^*_{s_2}(P,\succ)$, we have that $t_2 \succ_{s_2} t_1$; otherwise, $t_1$ would have a higher priority than a teacher in $DA^*_{s_2}(P,\succ)$ who is not an initial teacher of $s_2$, a contradiction of Lemma 1. Therefore, if $DA^*_{t_2}(P,\succ) = s_2 P_{t_2} \mu(t_2)$, then we would have $(t_2, s_2) \in B_{\mu}$, a contradiction to $B_{\mu} \subseteq B_{DA^*(P,\succ)}$.

Case 2. $t_2 = DA^*_{s_2}(P,\succ) \cap \mu_0(s_2)$.

In this case, if $DA^*_{t_2}(P,\succ) = \mu_0(t_2) = s_2 P_{t_2} \mu(t_2)$, then we would have the contradiction that $\mu$ is IR.

We can repeat the argument to show that $\forall \ell = 1, \ldots, n_k$, we must have $\mu(t_\ell) = s_{\ell+1} P_{t_\ell} s_\ell = DA^*_{t_\ell}(P,\succ)$ (with $n_k + 1 := 1$). Note that this result necessarily implies that $C_k$ is a cycle and not a chain.

Now, fix a set $C_k := \{(t_1, s_1), \ldots, (t_{n_k}, s_{n_k})\}$.

First, assume that $C_k$ is a cycle. In that case, all the nodes involve a teacher and a school. Without loss of generality, fix teacher $t_1$ and assume that $s_1 = DA^*_{t_1}(P,\succ) P_{t_1} s_2 = \mu(t_1)$. Note that, since $DA^*(P,\succ)$ and $\mu$ are IR, we have $s_1 \neq \mu_0(t_1)$. Consider teacher $t_{n_k}$ and assume that $s_1 = \mu(t_{n_k}) P_{n_k} DA^*_{t_{n_k}}(P,\succ) = s_{n_k}$. Since $t_1$ is assigned to $s_1$ under $DA^*(P,\succ)$, we trivially have that $(t_1, s_1) \notin B_{DA^*(P,\succ)}$. Since $t_1 \in DA^*_{s_1}(P,\succ) \setminus \mu_0(s_1)$, Lemma 1 implies that $t_1 \succ_{s_1} t_{n_k}$; otherwise, $t_{n_k}$ would have a higher priority in $s_1$ than a teacher assigned to it who was not an initial teacher of $s_1$, contradicting Lemma 1. Therefore, $(t_1, s_1) \in B_{\mu}$, a contradiction to $B_{\mu} \subseteq B_{DA^*(P,\succ)}$.

Assume now that $C_k$ is a chain that ends with the empty node, i.e., $(t_{n_k}, s_{n_k}) := (\emptyset, \emptyset)$. By construction, if $t := t_{n_k-1}$, $\mu(t) = \emptyset \neq DA^*_{t}(P,\succ)$. Since $DA^*$ is IR, we have that $DA^*_{t}(P,\succ) P_{\emptyset} = \mu(t)$. We can apply the same argument as before to show that $t_{n_k-2}$ must also be worse off and continue to show that all the teachers in the chain must be worse off under $\mu$.

Finally, assume that $C_k$ is a chain that ends with a non-empty node, i.e., $(t_{n_k}, s_{n_k}) = (\emptyset, s)$ for some integer $l$ and $s \in S$. Take teacher $t := t_{n_k-1}$. Note that it cannot be the case that $s_{n_k} = s P_{t} DA^*_{t}(P,\succ)$. Indeed, this implies that $|DA^*_{s}(P,\succ)| < q_s$, which would imply that $t$ is blocking $DA^*(P,\succ)$ with a school that does not fill all its positions, a contradiction to Lemma 1. Therefore, we must have that $DA^*_{t}(P,\succ) P_{t} \mu(t) = s$. The same argument as before applies, so we conclude that all the teachers in $C_k$ must be worse off under $\mu$.

Note that the previous proof implies that if the teachers in the set $C_k$ are all strictly better off under $\mu$, then this set must be a cycle. The next lemma tells us that if an IR matching $\mu$ has strictly less envy than $DA^*(P,\succ)$, i.e., $B_{\mu} \subset B_{DA^*(P,\succ)}$, then there must be a cycle in the decomposition from $DA^*(P,\succ)$ to $\mu$ that makes all the teachers strictly better off:

**Lemma 4.** Assume that a IR matching $\mu$ has strictly less envy than $DA^*(P,\succ)$, i.e., $B_{\mu} \subset B_{DA^*(P,\succ)}$. Then, in any decomposition given in Lemma 2, there must exist a cycle $C_k$ that respects the condition 1 of Lemma 3, i.e., that makes all its teachers strictly better off.

**Proof.** By contradiction, assume that all the cycles and chains make all their teachers strictly worse off, i.e., $\forall t \in T, DA^*_{t}(P,\succ) R_{t} \mu(t)$. Fix any pair $(t, s) \in B_{DA^*(P,\succ)}$. Lemma 1 implies that $|DA^*_{s}(P,\succ)| = q_s$ and there exists $t' \in DA^*_{s}(P,\succ) \cap \mu_0(s)$ s.t. $t \succ_{s} t'$. If $\mu(t') \neq DA^*_{t'}(P,\succ)$, then our assumption implies that $\mu(t') = DA^*_{t'}(P,\succ) P_{t'} \mu(t')$, contradicting that $\mu$ is IR. Therefore, $t' \in DA^*_{s}(P,\succ) \cap \mu(s)$ such that $(t, s) \in B_{\mu}$. We conclude that $B_{DA^*(P,\succ)} \subseteq B_{\mu}$, contradicting that $B_{\mu} \subset B_{DA^*(P,\succ)}$.

The next terminologies and lemmas are borrowed from Kesten and Kurino (2017). For a given profile $(P,\succ)$ and a matching $\mu$, we say that a school $s$ is:
- Overdemanded at $P$ under $\mu$: $\exists t \in T$ s.t. $s_P(t) \mu(t)$.
- Weakly-underdemanded at $P$ under $\mu$: $\forall t \in T: \mu(t) R_t s$.

**Lemma 5** (Lemma 1, Kesten and Kurino, 2017). For any profile $(P, \succ)$, there exists a weakly underdemanded school at $P$ under $DA^*(P, \succ)$.

The following terminology is also defined in Kesten and Kurino (2017): fix a preference profile $P_t$ for a teacher $t$ and two schools $s, s'$ s.t. $s_P(t) s'$. We say that a profile $P_t'$ upgrades $s'$ above $s$ in $P_t$ if i) $s' P_t s$, ii) there is no $s'' \in S$ s.t. $s' P_t s'' P_t s$ and iii) the relative ranking of all the other schools remains the same between $P_t'$ and $P_t$. Our proof will use the following technical lemma adapted to our teacher assignment setting:

**Lemma 6** (Lemma 1, Kesten and Kurino, 2017). Suppose that under $DA^*(P, \succ)$, teacher $t$ is assigned to school $s$ that is overdemanded at $P$ under $DA^*(P, \succ)$. Then, there exist $P_t'$ and a weakly underdemanded school $s'$ at $P$ under $DA^*(P, \succ)$ such that $P_t'$ upgrades $s'$ above $s$ in $P_t$, $DA_t^*(P_t', P_{-t}, \succ) = s'$, and $s'$ is weakly underdemanded at $(P_t', P_{-t})$ under $DA^*(P_t', P_{-t}, \succ)$.

It is easy to show that the lemma naturally applies to $DA^*$. Indeed, the definitions of overdemanded and weakly-underdemanded schools do not depend on the priority profile of the schools; $DA^*$ just modifies the priority profile of the schools and runs a standard DA over it. In our teacher assignment setting, the following lemma elicits an important property of an underdemanded school:

**Lemma 7.** If $\mu$ is an IR matching s.t. $B_\mu \subseteq B_{DA^*(P, \succ)}$ and $s := DA_t^*(P, \succ)$ is a weakly-underdemanded school at $P$ under $DA^*(P, \succ)$, then $DA_t^*(P, \succ) R_t \mu(t)$.

**Proof.** By contradiction, assume that $\mu(t) P_t DA_t^*(P, \succ)$. Since $\mu$ is IR and $\mu(t) \neq DA_t^*(P, \succ)$, Lemma 2 tells us that for any decomposition from $DA^*(P, \succ)$ to $\mu$, there exists a cycle $C_k := \{(t_1, s_1), \ldots, (t_n, s_n)\}$ and $\ell \in \{1, \ldots, n\}$ s.t. $t_\ell = t$. Since $\mu(t) P_t DA_t^*(P, \succ)$, Lemma 3 tells us that all the teachers in cycle $C_k$ must be strictly better-off under $\mu$ than under $DA^*(P, \succ)$. In particular, for teacher $t_{\ell-1}$, we have that $s = DA_t^*(P, \succ) = \mu(t_{\ell-1}) P_{t_{\ell-1}} DA_{t_{\ell-1}}^*(P, \succ)$, contradicting that $s$ was weakly-underdemanded at $P$ under $DA^*(P, \succ)$.

We now have all the elements to prove Proposition 1:

**Proof of Proposition 1.** Assume that there exists an IR mechanism $\varphi$ that has strictly less envy than $DA^*$. We will show that this mechanism cannot be strategy-proof.

If $\varphi$ has strictly less envy than $DA^*$, there must exists a profile $(P, \succ)$ s.t. $B_{\varphi(P, \succ)} \subseteq B_{DA^*(P, \succ)}$. By Lemma 2, there exists a decomposition $C_1, \ldots, C_k$ from $DA^*(P, \succ)$ to $\varphi(P, \succ)$. By Lemma 4, there exists one of these sets that is a cycle. W.l.o.g., say $C_1$ s.t. all the teachers involved in it are strictly better-off under $\varphi(P, \succ)$ than under $DA^*(P, \succ)$. Fix any teacher $t$ who is part of cycle $C_1$ and let $s := DA_t^*(P, \succ)$, $s' := \varphi_t(P, \succ)$ and $t'$ be the teacher of the node that points to $(t, s)$ so that $\varphi_{t'}(P, \succ) = s$. Since all the teachers in cycle $C_1$ are strictly better off, $s P_t DA_t^*(P, \succ)$ and $s$ is overdemanded at $P$ under $DA^*(P, \succ)$. Using Lemma 6, there exists a weakly-underdemanded school $s''$ at $P$ under $DA^*(P, \succ)$ and a preference profile $P_t'$ that upgrades $s''$ above $s$ in $P_t$ s.t. i) $DA_t^*(P_t', P_{-t}, \succ) = s''$ and ii) $s''$ is weakly-underdemanded at $(P_t', P_{-t})$ under $DA^*(P_t', P_{-t}, \succ)$.

According to ii), since $s''$ is weakly-underdemanded at $(P_t', P_{-t})$ under $DA^*(P_t', P_{-t}, \succ)$ and by the assumption $B_{\varphi(P_t', P_{-t}, \succ)} \subseteq B_{DA^*(P_t', P_{-t}, \succ)}$, Lemma 7 implies that:

$$s'' := DA_t^*(P_t', P_{-t}, \succ) R_t \varphi_t(P_t', P_{-t}, \succ)$$

However, since $s' = \varphi_t(P, \succ) P_t DA_t^*(P, \succ) = s$ and $P_t'$ upgrades $s''$ above $s$, then $s''$ is ranked immediately before $s$ and $s'$ is still strictly preferred to $s''$ under $P_t'$:

$$s' = \varphi_t(P_t', P_{-t}, \succ) P_t' s'' := DA_t^*(P_t', P_{-t}, \succ) R_t \varphi_t(P_t', P_{-t}, \succ)$$

Therefore, we conclude that $\varphi$ is not strategy-proof. □
6 Discussions

In the previous section, we showed that there are other mechanisms than DA∗ (resp. TTC∗) that are minimal envy mechanisms in the set of IR and SP (resp. IR, SP and PE) mechanisms. However, as our examples showed, their constructions are quite specific and select a particular mechanism for a particular priority profile. A legitimate question is whether one can find other natural mechanisms than DA∗ or TTC∗ that are minimal envy mechanisms in some well defined sets.

In considering the assignment of teachers to schools, Combe, Tercieux, and Terrier (2016) considered another mechanism: the Teacher-Optimal Block Exchange algorithm (TO-BE). They considered a many-to-one environment with only tenured teachers, i.e., n = 0. To study minimal envy, because their mechanism shares similar features as those of the TTC mechanism, consider a one-to-one setting. Their motivation is twofold:

1. They note that the DA∗ mechanism is not efficient in a strong sense: one can reassign teachers to schools such that both teachers and schools obtain an assignment that they rank higher. They refer to a matching/mechanism not suffering from this problem as two-sided Pareto-efficient (2-PE).²⁹

2. Even if schools are not strategic entities and their priorities are fixed by law, they argue that those priorities incorporate some welfare objectives that the policy maker would like to achieve. For instance, in France, those priorities incorporate points regarding the experience of the teachers or their ability to move closer to their partner. Thus, they consider a 2-sided notion of individual rationality: 2-Individual Rationality (2-IR). A matching µ is 2-IR if ∀t ∈ T, µ(t) ⪰ µ0(t) and ∀s ∈ S, µ(s) ⪰ s µ0(s), so teachers and schools obtain a weakly better assignment.

The TO-BE mechanism is a 2-IR, 2-PE and SP mechanism. Using data on the assignment of teachers in France, they show that this mechanism performs better than the DA∗ algorithm in terms of the movement and welfare of teachers. They also note that the 2-IR property allows the achievement of better policy objectives with respect to the distribution of teachers compared to those of other IR (but not 2-IR) mechanisms, such as TTC.³⁰ The TO-BE mechanism works as follows: first, for S′ ⊆ S, a matching µ and a teacher t, define the opportunity set of teacher t as Opp(t, S′) := {s ∈ S′ : t ⪰ s µ(s′)}, i.e., the schools in S′ that give t a weakly higher priority than their initial teacher. Then:

\begin{align*}
\text{Step 0.} & \quad \text{Let } S(0) = S, T(0) = T \text{ and } \mu(0) = \mu_0. \\
\text{Step } k \geq 1 & \quad \text{Build the graph } (N, E) \text{ s.t. } N = \{(t, \mu_0(t)) : t \in T(k - 1)\}. \text{ For a node } (t, s) \in N, \text{ let } (t, s) \text{ point to } (t′, s′) \text{ if } s′ \text{ is the favorite school of } t \text{ in } \text{Opp}(t, S(k - 1)).³¹ \text{ There will be cycles in this graph, and all the cycles are disjoint. Implement a cycle}³² \text{ in matching each teacher to the school of the node his/her node is pointing to. Define } T(k) \text{ and } S(k) \text{ in deleting the assigned teachers and schools. If these sets are empty, stop the algorithm; otherwise, go to step } k + 1.
\end{align*}

For some profile (P, ≻), BTO-BE(P,≻) ⊂ BDA∗(P,≻). However, as for TTC∗, DA∗ and TO-BE are not comparable according to the less envy relation.³³ In addition, one can also easily show that TTC∗ and TO-BE are not comparable according to the former relation. Since TO-BE is 2-PE, it is not in the set

²⁹Our previous notion of Pareto-efficiency only considered welfare improvement on the teachers’ side.
³⁰We refer the interested reader to the cited article for further details. In a one-to-one framework as we are considering here, the TO-BE mechanism and the 2S-TTC, proposed by Dur and Unver (2015) to study the Tuition Exchange Programs such as Erasmus, are equivalent.
³¹Their original definition is slightly more general since they consider a many-to-one setting. Note that, at any step k, for any teacher t ∈ T(k − 1), µ0(t) ∈ Opp(t, S(k − 1)) so that a node (t, s) can point to itself, forming a (self-)cycle
³²The order in which the cycles are implemented does not influence the final matching, so if there are many cycles, the selection of which one to implement does not matter.
³³Note that in Example 1, TTC∗ and TO-BE lead to the same matching.
of IR, SP and PE mechanisms that we considered for TTC*. Note that TO-BE is a 2-IR mechanism. Thus, the natural class of mechanisms to consider is the class of 2-IR and SP mechanisms rather than, as for DA*, the class of IR and SP mechanisms. In the latter, one can easily exhibit mechanisms that have strictly less envy than TO-BE, so the latter is not a minimal envy mechanism in this set:

**Proposition 4.** There are mechanisms that are IR, SP and have strictly less envy than TO-BE.

*Proof.* Consider the following simple environment with only 2 teachers, $t_1$ and $t_2$, initially assigned to, respectively, $s_1$ and $s_2$. In this setting, the standard Shapley-Scarf TTC mechanism is IR and SP and trivially leads to fewer blocking pairs than TO-BE since there is only one possible exchange. Under TO-BE, if one of the two teachers has a lower priority in the school of the other one, then TO-BE does not implement the exchange. On can define a mechanism $\varphi$ that is the Shapley Scarf TTC mechanism when there are only two teachers and is TO-BE otherwise. It can be trivially verified that such a mechanism is IR and SP and has strictly less envy than TO-BE.

The following proposition shows that, in a one-to-one environment with only tenured teachers ($n = 0$) and in the class of 2-IR and SP mechanisms, there is indeed no mechanism with strictly less envy than TO-BE, so TO-BE is a minimal envy mechanism in this set.

**Proposition 5.** Consider a one-to-one setting with only tenured teachers ($n = 0$); then, TO-BE is a minimal envy mechanism in the set of 2-IR and SP mechanisms.

*Proof.* The proof is relegated to Section B of the Appendix.

Combe, Tercieux, and Terrier (2016) showed that, in a one-to-one setting with $n = 0$, TO-BE is the unique 2-IR, 2-PE and SP. However, the result of Proposition 4 is stronger since it considers the set of all the 2-IR and SP mechanisms and not just the set of 2-IR, SP and 2-PE mechanisms. This is different from the TTC* mechanism, which is not a minimal envy mechanism if one considers the set of IR and SP mechanisms, even if there are only tenured teachers. If one relaxes the 2-PE property, then it is possible to find other minimal envy mechanisms in the set of 2-IR and SP mechanisms. Take the same setting as in Example 1. Define the mechanism $\varphi$ as follows: if the priority profile is $\succ$, as in Example 1, then $\forall P$, let $\varphi(P, \succ) = \text{TO-BE}(P, \succ)$, where $\succ := (\succ_{s_1}, \succ_{s_2}, \succ_{s_3})$ with $\succ_{s_2} : t_3, t_2, t_1$, i.e., we artificially modify $s_2$’s priority profile in ranking $t_1$ below $t_2$. If the priority profile differs from $\succ$ or the number of teachers differs, let $\varphi$ be equal to TO-BE. Note that, in profile $(P, \succ)$ of Example 1, $\varphi(P, \succ) = \text{DA}^*(P, \succ)$ and TO-BE$(P, \succ) = \text{TTC}^*(P, \succ)$, so $B_{\varphi(P, \succ)} = B_{\text{DA}^*(P, \succ)} \neq B_{\text{TO-BE}(P, \succ)}$ and $\varphi$ and TO-BE cannot be compared using the “strictly less envy” relation. Note that, by construction, $\varphi$ is 2-IR and also trivially SP.\footnote{The modification of the priority ordering of $s_2$ in profile $(P, \succ)$ does not depend on the reported preferences of the teachers and the mechanism used is TO-BE, so strategy-proofness is satisfied.} One can also easily verify that it is a minimal envy mechanism in the set of 2-IR and SP mechanisms.

Similarly to TTC*, if one considers a many-to-one setting with only tenured teachers, then one can find a 2-IR, 2-PE and SP mechanism that has strictly less envy than TO-BE.\footnote{Since the statement follows naturally, we omit the example here but can provide it upon request.} However, if one considers an one-to-one intermediate setting with $0 < n < N$, then it is still an open question whether we can naturally extend the TO-BE mechanism to maintain its minimal-envyproperty in the set of 2-IR, 2-PE and SP mechanisms. The main difficulty is that the 2-IR property imposes that a school with an initially assigned teacher cannot end up being vacant. If one assumes that $\forall s \in S$, $\emptyset \preceq s$, then one can naturally extend the TO-BE algorithm in letting vacant schools\footnote{Nodes of the form $(\emptyset, s)$ in the graph of TO-BE if $\mu_0(s) = \emptyset$. Initially unassigned teachers would appear in nodes of the form $(t, \emptyset)$.} point to the node of their highest-ranked teacher in each step. In doing so, as in TTC*, a school with an initially assigned teacher can end up being vacant, but the results would still respect the 2-IR definition under the assumption of priorities. One can use a
similar proof as that of Proposition 3 and Proposition 4 to show that TO-BE would indeed be a minimal envy mechanism in the set of 2-IR, 2-PE and SP mechanisms.

However, if $\forall s \in S, \mu_0(s) \succeq s$, then a 2-IR matching cannot leave a school with an initially assigned teacher vacant. Notably, if the highest-ranked teacher of a vacant school is initially assigned to another school, then letting the former school point to such a teacher, as in TTC*, can leave the latter school vacant. Therefore, if the vacant schools are no longer pointing to their highest-ranked teacher, the techniques used in the proof of Proposition 3 and 4 cannot be used. In that case, the question of whether one can naturally extend the TO-BE algorithm to properly define the pointing behavior for vacant schools to maintain the minimal envy property remains open and is left for future research.

7 Conclusion

Our paper showed that two of the main individually rational (IR) mechanisms, the DA* and the TTC*, used to assign teachers to schools when some teachers may be initially assigned to a school are minimal envy mechanisms. For the former, we showed that it is not possible to find another IR and strategy-proof (SP) mechanism that always leads to fewer blocking pairs (in the setwise inclusion sense). For TTC*, when schools only have one seat, we showed that one cannot find another IR, SP and Pareto-efficient mechanism that always results in fewer blocking pairs.

A special case of our setting is the standard school choice framework, i.e., all the agents are initially unassigned, and the two mechanisms that we consider, DA* and TTC*, both collapse to the school DA and TTC mechanisms in that setting. Therefore, our results extend two important results of the literature on school choice to a more general model where i) the DA mechanism is, trivially, a minimal envy mechanism in the set SP mechanisms and ii) the TTC algorithm is a minimal envy mechanism in the set of SP and PE mechanisms.

Finally, we discussed the results concerning another mechanism in the literature: the TO-BE mechanism. The latter mechanism is individually rational for teachers and schools (2-IR), is SP and respects a stronger form of efficiency, i.e., 2-Pareto-efficiency (2-PE). In a one-to-one framework, where all the agents are initially assigned, we show that TO-BE is a minimal envy mechanism in the set of 2-IR and SP mechanisms. However, there remains an open question of how to naturally extend the mechanism to incorporate, in a one-to-one setting, possible agents without any initial assignment while maintaining the minimal envy property.

References


APPENDIX

A Proof of Proposition 3

We will prove a slightly stronger claim: any IR, SP and PE mechanism $\varphi$ that has less envy than TTC* must be equivalent to the latter. Assume that there exists another IR, SP and PE mechanism $\varphi$ and a preference profile $(P,) $ s.t. $\varphi(P,) \neq \text{TTC}^*(P,) $ and $B_{\varphi(P,)} \subseteq B_{\text{TTC}^*(P,)}$. We will show that $\varphi(P,) = \text{TTC}^*(P,) $. The proof follows a similar strategy to that of Abdulkadiroglu, Che, Pathak, Roth, and Tercieux (2017) for the TTC mechanisms in a school choice setting and uses an induction over the steps of TTC*.

Consider the first step of $\text{TTC}^*(P,)$ and a teacher $t$ assigned at that step. We will show that $\text{TTC}^*_0(P,) = \varphi(P,)$. Fix the cycle $C$ with which teacher $t$ has been assigned and let $t := t_1$ in $C := \{t_1, s_1, t_2, s_2, \ldots, t_K, s_K\}$.

Assume that $s := \varphi_t(P,) \neq \text{TTC}^*_n(P,) = s_1$. In this step, $t$ is pointing to his/her favorite school so that necessarily $s_1 P_t s$. There are two cases to consider:

Case 1: $s_K = \mu_0(t_1)$. Assume that $t_1$ reports the profile $P'_t : s_1, s_K$. Note that, with this new profile, $\text{TTC}^*$ is the same so that $\text{TTC}^*(P'_t, P_{-t_1},,) = \text{TTC}^*(P,)$ and, in particular, $s_1 = \text{TTC}^*_1(P'_t, P_{-t_1},,)$. Note that, since $\varphi$ is SP and $s_1 P_t s$, then $\varphi_t(P'_t, P_{-t_1},,) \neq s_1$. However, since $\varphi$ is IR and $s_K = \mu_0(t_1)$, it must be the case that $\varphi_t(P'_t, P_{-t_1},,) = s_K$.

Case 2: $s_K \neq \mu_0(t_1)$. In the first step of $\text{TTC}^*$, schools point to their first-ranked teacher according to $\succ$, the modified priority profile where schools with an initial teacher rank him/her at the top of their profile. Since $s_K \neq \mu_0(t_1)$ but $s_K$ is pointing to $t_1$, we have that $\varphi_t(P'_t, P_{-t_1},,) \neq \emptyset$. Therefore, $\varphi_t(P'_t, P_{-t_1},,)$ and teacher $t_1$ is the highest-ranked teacher in the priority profile $\succ_s$. Assume $t_1$ reports the profile $P'_t : s_1, s_K, \mu_0(t_1)$. Again, $\text{TTC}^*$ is the same so $\text{TTC}^*(P'_t, P_{-t_1},,) = \text{TTC}^*(P,)$ and, in particular, $s_1 = \text{TTC}^*_1(P'_t, P_{-t_1},,)$. Note that since $\varphi$ is SP and $s_1 P_t s$, then $\varphi_t(P'_t, P_{-t_1},,) \neq s_1$, and since $\varphi$ is IR, $\varphi_t(P'_t, P_{-t_1},,) \in \{s_K, \mu_0(t_1)\}$. Since $s_1 P_t s_K$, we trivially have that $(t_1, s_K) \notin B_{\text{TTC}^*(P'_t, P_{-t_1},,)}$. If $\varphi_t(P'_t, P_{-t_1},,) = \mu_0(t_1)$, then since $t_1$ is the highest-ranked teacher in $\succ_s$, we would have $(t_1, s_K) \in B_{\varphi_t(P'_t, P_{-t_1},,)}$, a contradiction to $B_{\varphi_t(P'_t, P_{-t_1},,)} \subseteq B_{\text{TTC}^*(P'_t, P_{-t_1},,)}$. Therefore, we conclude that $\varphi_t(P'_t, P_{-t_1},,) = s_K$.

Let $P'_t$ be the preference profile of $t_1$ under one of the two above cases, depending on which $s_K = \mu_0(t_1)$. In both cases, we have seen that $\varphi_t(P'_t, P_{-t_1},,) = s_K$. Since we are in a one-to-one framework, $s_K = \text{TTC}^*_k(P'_t, P_{-t_1},,) \neq \varphi_t(P'_t, P_{-t_1},,)$. Again, since teachers point to their favorite school in the first step of $\text{TTC}^*$, we have:

$$s_K = \text{TTC}^*_k(P'_t, P_{-t_1},,) \varphi_t(P'_t, P_{-t_1},,)$$

Similarly as before, let $t_K$ be profile $P'_t : s_K, s_{K-1} = \mu_0(t_K)$ or $P'_t : s_K, s_{K-1}, \mu_0(t_K)$ if $s_{K-1} \neq \mu_0(t_K)$. Using a similar argument as that for teacher $t_1$, one can show that:

$$\text{TTC}^*(P'_t, P_{t_K}, P_{-\{t_1,t_K\},,}) = \text{TTC}^*(P'_t, P_{-t_1},,)$$

By repeating the argument recursively for $t_{K-1}$ and other teachers in $C$, we obtain the following: $\forall k = 1, \ldots, K, \varphi_k(P',,) = s_{k-1}$ with $0 := K$ with $P'_t : s_k, s_{k-1}$ if $\mu_0(t_k) = s_{k-1}$ or $P'_t : s_k, s_{k-1}, \mu_0(t_k)$ if $\mu_0(t_k) \neq s_{k-1}$. Note that this result contradicts that $\varphi$ is PE since all the teachers in $C$ can now exchange their assignment under $\varphi(P',,)$ and be strictly better off. Therefore, we have shown that, for any profile

Remember that we are in a one-to-one setting so that schools can only have one initial teacher.

Note that it can be the case that $\mu_0(t_1) = \emptyset$.
(P, \succ), all the teachers assigned at the first step of \( \text{TTC}^*(P, \succ) \) must have the same assignment between \( \text{TTC}^*(P, \succ) \) and \( \varphi(P, \succ) \).

Now, assume that for any profile \((P, \succ)\), all the teachers assigned up to step \(k-1\) of \( \text{TTC}^*(P, \succ) \) have the same assignment between \( \text{TTC}^*(P, \succ) \) and \( \varphi(P, \succ) \).

Consider a teacher \( t \) assigned in step \( k \) of \( \text{TTC}^*(P, \succ) \). Let \( t_1 := t \) and \( C := \{t_1, s_1, t_2, s_2, \ldots, t_K, s_K\} \) be the cycle of step \( k \) \( \text{TTC}^*(P, \succ) \) that assigned teacher \( t \) so that \( \text{TTC}^*(P, \succ)_{t_1} = s_1 \). Assume that \( s := \varphi(P, \succ)_{t_1} \neq s_1 \). Using our induction hypothesis, note that school \( s \) cannot be a school that under \( \text{TTC}^*(P, \succ) \) has been assigned at a step \( k' < k \). Indeed, if this was the case, then the teacher assigned to \( s \) at that step would still be matched at \( s \) under \( \varphi(P, \succ) \). Since we are in a one-to-one setting, it would contradict that \( t_1 \) was assigned to it. Therefore, in step \( k \) of \( \text{TTC}^*(P, \succ) \), school \( s \) is still in the graph and has not been assigned. Since in that step, teacher \( t_1 \) points to \( s_1 \), his/her favorite school among those still available, we have that \( s_1 P_1 s \). As before, there are two cases to consider:

Case 1: \( s_K = \mu_0(t_1) \). Assume that \( t_1 \) reports the profile \( P'_t : s_1, s_K, \mu_0(t_1) \). Note that with this new profile, \( \text{TTC}^* \) is the same, so \( \text{TTC}^*(P'_t, P_{t-1}, \succ) = \text{TTC}^*(P, \succ) \) and, in particular, \( s_1 = \text{TTC}^*_t(P'_t, P_{t-1}, \succ) \). Note that since \( \varphi \) is SP and \( s_1 P_t s, \varphi(P'_t, P_{t-1}, \succ) \neq s_1 \). However, since \( \varphi \) is IR and \( s_K = \mu_0(t_1) \), it must be the case that \( \varphi_t(P'_t, P_{t-1}, \succ) = s_K \).

Case 2: \( s_K \neq \mu_0(t_1) \). Assume \( t_1 \) reports the profile \( P'_t : s_1, s_K, \mu_0(t_1) \). Again, \( \text{TTC}^* \) is the same, so \( \text{TTC}^*(P'_t, P_{t-1}, \succ) = \text{TTC}^*(P, \succ) \) and, in particular, \( s_1 = \text{TTC}^*_t(P'_t, P_{t-1}, \succ) \). Additionally, note that all the teachers assigned before step \( k \) of \( \text{TTC}^*(P, \succ) \) are the same as those assigned before step \( k \) of \( \text{TTC}^*(P'_t, P_{t-1}, \succ) \). Since \( s_1 P_t s, K \), we trivially have that \( (t_1, s_K) \notin B_{\text{TTC}^*}(P'_t, P_{t-1}, \succ) \), and since \( \varphi \) has less envy than \( \text{TTC}^* \), it must be the case that \( (t_1, s_K) \notin B_{\varphi(P'_t, P_{t-1}, \succ)} \). Note that since \( \varphi \) is SP and \( s_1 P_t s, \varphi(P'_t, P_{t-1}, \succ) \neq s_1 \), and since \( \varphi \) is IR, \( \varphi(P'_t, P_{t-1}, \succ) \in \{s_K, \mu_0(t_1)\} \). By construction of \( \text{TTC}^* \), for all the teachers \( t' \neq t_1 \) who have not been assigned yet at step \( k \) of \( \text{TTC}^*(P'_t, P_{t-1}, \succ) \) and since \( s_K \) points to \( s_1 \), we have that \( t_1 \sim_{s_K} t' \). If \( \mu_0(s_K) = \emptyset \), then the modified priority ordering \( \succ_{s_K} \) can be changed into the true \( \succ_{s_K} \). Thus \( \mu_0(s_K) = \emptyset \), note that since \( s_K \neq \mu_0(t_1) \) and \( s_K \) was pointing to \( t_1 \neq \tilde{t} \) at step \( k \), it must be the case that \( \tilde{t} \) has been assigned at a step \( k' \neq k \). Since the priority ordering \( \succ_{s_K} \) maintains the same relative ranking as \( \succ_{s_K} \) between teachers who are not the initial teachers of \( s_K \), we also have that \( t_1 \succ_{s_K} t' \) for any teacher \( t' \) who has not been assigned yet at the beginning of step \( k \) of \( \text{TTC}^*(P'_t, P_{t-1}, \succ) \). Now, note that teacher \( t' := \varphi_{s_K}(P'_t, P_{t-1}, \succ) \) has not been assigned yet at the beginning of step \( k \) of \( \text{TTC}^*(P'_t, P_{t-1}, \succ) \). Indeed, if this was not the case, our induction hypothesis would imply that:

\[ \varphi_{t'}(P'_t, P_{t-1}, \succ) = \text{TTC}^*_v(P'_t, P_{t-1}, \succ) = s_K \]

Therefore, \( t' = t_K \), implying that \( t_K \) was assigned to a step \( k' < k \) of \( \text{TTC}^*(P'_t, P_{t-1}, \succ) \), a contradiction. \( t' \) is still present at the beginning of step \( k \) of \( \text{TTC}^*(P'_t, P_{t-1}, \succ) \) and \( t_1 \succ_{s_K} t' \). If \( \varphi_t(P'_t, P_{t-1}, \succ) \neq s_K \) (so \( t_1 \neq t' \)), then \( \varphi_t(P'_t, P_{t-1}, \succ) = \mu_0(t_1) \), and since \( s_K P_t \mu_0(t_1) \) and \( t_1 \succ_{s_K} t' = \varphi_{s_K}(P'_t, P_{t-1}, \succ) \), we have that \( (t_1, s_K) \notin B_{\varphi(P'_t, P_{t-1}, \succ)} \), a contradiction. Therefore, we conclude that \( \varphi_t(P'_t, P_{t-1}, \succ) = s_K \).

Let \( P'_t \), be one of the two above profiles depending on whether we are in Case 1 or Case 2. In both cases, \( \varphi_t(P'_t, P_{t-1}, \succ) = s_K \). Since we are in a one-to-one setting, \( \text{TTC}^*_K(P'_t, P_{t-1}, \succ) = s_K \neq \varphi_{s_K}(P'_t, P_{t-1}, \succ) := s \). Using our induction hypothesis, \( s \) cannot be a school that has been assigned before step \( k \) of \( \text{TTC}^*_K(P'_t, P_{t-1}, \succ) \), so school \( s \) must have been available to \( t_K \) at the beginning of step \( k \). Since in each step, teachers are pointing to their favorite school among those remaining, we conclude that \( s_K P_t s \). As before, let \( t_K \) be the profile \( P'_t : s_K, s_K, \mu_0(t_K) \) if \( s_K = \mu_0(t_K) \) or \( P'_t := s_K, s_K, \mu_0(t_K) \).

\[39\] If \( s = \mu_0(t_1) \), it can be the case that \( P'_t = P_t \). However, the same contradiction as the one shown below will hold and we would still conclude that \( \varphi_t(P'_t, P_{t-1}, \succ) = s_K \).
if \( s_{k-1} \neq \mu_0(t_K) \). With similar arguments as those for \( t_1 \), we have that:

\[
\text{TTC}^*(P'_t, P'_t, P_{-\{t_1, t_K\}}, \succ) = \text{TTC}^*(P'_t, P_{-t_1}, \succ)
\]

\[
\varphi_{t_K}(P'_t, P'_t, P_{-\{t_1, t_K\}}, \succ) = s_{k-1}
\]

By repeating the argument recursively for \( t_{k-1} \) and other teachers in \( C \), we conclude that \( \forall k = 1, \ldots, K, \varphi_{t_k}(P'_t, \succ) = s_{k-1} \) with \( P'_{t_k} : s_k, s_{k-1} \) if \( \mu_0(t_k) = s_{k-1} \) or \( P'_{t_k} : s_k, s_{k-1}, \mu_0(t_k) \) if \( \mu_0(t_k) \neq s_{k-1} \). Again, this contradicts that \( \varphi \) is PE since at profile \( P' \), all the teachers in \( C \) can exchange their allocation under \( \varphi \) and be strictly better off.

Therefore, we conclude that if a mechanism is IR, SP and PE and has less envy than \( \text{TTC}^* \), then the two must be equivalent such that \( \text{TTC}^* \) is indeed a minimal envy mechanism in the set of IR, SP and PE mechanisms.

\section{Proof of Proposition 5}

Let \( \varphi \) be a 2-IR and SP mechanism. Fix \((P, \succ)\) s.t. \( \varphi(P, \succ) \neq \text{TO-BE}(P, \succ) \). The proof is by induction over the steps of \( \text{TO-BE}(P, \succ) \).

Let \( C_1 := \{(t_1, s_1), \ldots, (t_K, s_K)\} \) be the cycle implemented at the first step of \( \text{TO-BE} \). Note that, by construction of \( \text{TO-BE} \), we have that \( t_K \succ s_1, t_1 \) so that \( (t_K, s_1) \in B_{\mu_0} \) and since \( s_1 = \text{TO-BE}_{t_K}(P, \succ), (t_K, s_1) \notin B_{\text{TO-BE}(P, \succ)} \). Assume that \( C_1 \) is not implemented by \( \varphi \). There is at least one teacher in the cycle \( C_1 \), w.l.o.g. say \( t_1 \), who is assigned to a different school than the one he/she is pointing to in the cycle \( C_1 \), i.e. \( s := \varphi_{t_1}(P, \succ) \neq \text{TO-BE}_{t_1}(P, \succ) = s_2 \). Since \( \varphi \) is 2-IR, we must have that \( t_1 \succ s, \mu_0(s) \) so that \( s \in \text{Opp}(t, S) \). Since, by definition of \( \text{TO-BE} \), \( t_1 \) is pointing to his/her favorite school in his/her opportunity set, we have that \( s_2 P_t s = \varphi_{t_1}(P, \succ) \). Now, assume that \( t_1 \) reports the following profile: \( P'_t : s_2, s_1 \). First, note that it does not change the assignment of \( \text{TO-BE} \) so that \( \text{TO-BE}(P'_t, P_{-t_1}, \succ) = \text{TO-BE}(P, \succ) \) and, in particular, \( (t_K, s_1) \notin B_{\text{TO-BE}(P'_t, P_{-t_1}, \succ)} \). But since \( \varphi \) is SP, we must have \( \varphi_{t_1}(P'_t, P_{-t_1}, \succ) = s_1 \). Again, since \( \varphi \) is 2-IR, let \( s' := \varphi_{t_k}(P'_t, P_{-t_1}, \succ) \), we must have that \( t_K \succ s', \mu_0(s') \) so that \( s' \in \text{Opp}(t, S(0)) \) and since \( t_K \) was pointing to his/her favorite school in this set, \( s_1 P_{t_k} s' \) so that \( (t_K, s_1) \in B_{\varphi(P'_t, P_{-t_1}, \succ)} \). A contradiction to \( B_{\varphi(P'_t, P_{-t_1}, \succ)} \subseteq B_{\text{TO-BE}(P'_t, P_{-t_1}, \succ)} \).

So we conclude that all the teachers assigned by the cycle \( C_1 \) must have the same assignment between \( \text{TO-BE}(P, \succ) \) and \( \varphi(P, \succ) \).\(^{40}\)

If one considers any profile \( P' \) s.t. \( \forall t \in T, \forall s \in S, \) we have \( sR_t \text{TO-BE}_t(P, \succ) = sR_t \text{TO-BE}_t(P, \succ) \), then \( \text{TO-BE}(P, \succ) = \text{TO-BE}(P', \succ) \). Indeed, all the teachers rank weakly higher the school that they obtain under \( \text{TO-BE}(P, \succ) \) so that the same cycles form under \( \text{TO-BE}(P', \succ) \). So for the teachers in \( C_1 \), which is the cycle of the first step of \( \text{TO-BE}(P, \succ) \), they can still be assigned with the same cycle \( C_1 \) at the step 1 of \( \text{TO-BE}(P', \succ) \). The same argument can be applied to show that these teachers must have the same assignment between \( \text{TO-BE}(P', \succ) = \text{TO-BE}(P, \succ) \) and \( \varphi(P', \succ) \).\(^{41}\) So our induction hypothesis is the following: assume that all the teachers assigned up to the step \( k - 1 \) of \( \text{TO-BE}(P, \succ) \) have the same assignment between \( \text{TO-BE}(P', \succ) \) and \( \varphi(P', \succ) \) for any \( P' \) s.t. \( \forall t \in T, \) we have \( sP_t \text{TO-BE}_t(P, \succ) = sP_t \text{TO-BE}_t(P', \succ) \).\(^{42}\)

---

\(^{40}\)Note that if the cycle \( C_1 \) is a self-cycle, i.e. \( K = 1 \). Then there is no 2-IR matching that assigns \( t_1 \) to a different school than \( s_1 \) so that the same conclusion holds.

\(^{41}\)Note that, under \( \text{TO-BE}(P, \succ) \), there might be additional new cycles at the step 1 of \( \text{TO-BE}(P', \succ) \) compared to the step 1 of \( \text{TO-BE}(P, \succ) \). However, the cycle \( C_1 \) will still remain at the step 1 of \( \text{TO-BE}(P', \succ) \) so that the claim concerning the teachers in \( C_1 \) is true.

\(^{42}\)Once again, we do not claim that a cycle implemented at the step \( k' < k \) of \( \text{TO-BE}(P', \succ) \) appears only at the step \( k' \) of \( \text{TO-BE}(P', \succ) \). However, since all the teachers rank weakly higher their assignment under \( \text{TO-BE}(P, \succ) \) in the profile \( P' \) compared to \( P \), we can always pick the cycles appearing in \( \text{TO-BE}(P', \succ) \) in an order that will indeed assign at the step \( k' \) of \( \text{TO-BE}(P', \succ) \) the teachers assigned at the step \( k' \) of \( \text{TO-BE}(P, \succ) \). Since the order in which we implement several existing cycles at a given step of the \( \text{TO-BE} \) algorithm does not influence its final outcome, our induction using the steps of \( \text{TO-BE}(P, \succ) \) is valid. The only effect of \( P' \) is to make cycles appear earlier in \( \text{TO-BE}(P', \succ) \) compared to \( \text{TO-BE}(P, \succ) \).
Let \( C_k := \{(t_1,s_1),\ldots,(t_K,s_K)\} \) be the cycle implemented at the step \( k \) of \( \text{TO-BE}(P,\succ) \). Assume that there is a teacher in this cycle, say \( t_1 \) who is assigned to a different school than the one he/she is pointing to under \( C_k \), i.e. \( s := \varphi_{t_1}(P,\succ) \neq \text{TO-BE}_{t_1}(P,\succ) = s_2 \). Since \( \varphi \) is 2-IR, we must have that \( t_1 \succ s \mu_0(s) \). Using our induction hypothesis, note that all schools assigned before the step \( k \) of \( \text{TO-BE} \) are assigned to the same teachers under both \( \varphi(P,\succ) \) and \( \text{TO-BE}(P,\succ) \). Thus, since \( s = \varphi_t(P,\succ) \), we must have that \( s \in \text{Opp}(t,S(k-1)) \). Since, at the step \( k \) of \( \text{TO-BE} \), teacher \( t \) is pointing to the node with his/her favorite school in \( \text{Opp}(t,S(k-1)) \), we have that \( t \succ s \mu_0(s) \). Using our induction hypothesis, note that all schools assigned before the step \( k \) of \( \text{TO-BE} \) are assigned to the same teachers under both \( \varphi(P,\succ) \) and \( \text{TO-BE}(P,\succ) \). Thus, since \( s = \varphi_{t_1}(P,\succ) \), we must have that \( s \in \text{Opp}(t,s_1) \). Since \( \varphi \) is 2-IR, let \( s' := \varphi_{t_K}(P_{t_1}',P_{t_1} - t_1,\succ) \neq s_1 \), we must have that \( t_K \succ s' \mu_0(s') \). If \( s_{t_K}P_{t_K}s' \), then \( (s_{t_K},s_1) \in B_{\varphi(P_{t_1}',P_{t_1} - t_1,\succ)} \). But since \( (t_K,s_1) \notin B_{\text{TO-BE}(P,\succ)} = B_{\text{TO-BE}(P_{t_1}',P_{t_1} - t_1,\succ)} \), it contradicts that \( B_{\varphi(P_{t_1}',P_{t_1} - t_1,\succ)} \subseteq B_{\text{TO-BE}(P_{t_1}',P_{t_1} - t_1,\succ)} \).

If \( s'P_{t_K}s_{t_1} \), then it means that \( s' \) was assigned at a step \( k' < k \) of \( \text{TO-BE}(P,\succ) \). Now, note that the profile \( (P_{t_1}',P_{t_1} - t_1,\succ) \) trivially respects the condition of our induction hypothesis. Using the latter, all the schools assigned at a step \( k' < k \) of \( \text{TO-BE}(P,\succ) \) have the same assignment between \( \varphi(P_{t_1}',P_{t_1} - t_1,\succ) \) and \( \text{TO-BE}(P_{t_1}',P_{t_1} - t_1,\succ) = \text{TO-BE}(P,\succ) \), contradicting that \( s' \) is the school assigned to \( t_K \) under \( \varphi(P_{t_1}',P_{t_1} - t_1,\succ) \).

So we conclude that \( \text{TO-BE}(P,\succ) = \varphi(P,\succ) \), a contradiction to \( \varphi(P,\succ) \neq \text{TO-BE}(P,\succ) \).